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Phase transitions and continuity properties of some random multifractal measures

Christian Webb

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Department of Mathematics and Statistics, Faculty of Science, University of Helsinki

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In this thesis

In addition to an introductory part, this thesis contains the following articles:

[i] Christian Webb: Exact asymptotics of the freezing transitions of a logarithmically correlated random energy model. J. Stat. Phys, 145 (2011), 1595–1619

[ii] Julien Barral, Antti Kupiainen, Miika Nikula, Eero Saksman, Christian Webb: Critical Mandelbrot Cascades (2012) http://arxiv.org/abs/1206.5444

[iii] Julien Barral, Antti Kupiainen, Miika Nikula, Eero Saksman, Christian Webb: Basic properties of critical lognormal multiplicative chaos (2013) http://arxiv.org/abs/1303.4548

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Introduction

Contents

CHAPTER 1

Motivation

This thesis is mainly about the existence and properties of random measures whose heuristic definition is

$$\mu_\beta(dx) = e^{\beta X(x)}dx, \tag{1.1}$$

where $\beta \geq 0$ is a parameter, X is a centered Gaussian random field on $\mathbb{R}^n$ with logarithmic correlations: as $x \to y$, $\mathbb{E}(X(x)X(y)) \sim -\log d(x,y)$, where d is either the ultrametric distance on $[0,1]^n$ or the Euclidean one, and dx is the Lebesgue measure on $\mathbb{R}^n$. With such a divergence in the variance of the field, it turns out that X can't be interpreted as a random function. The remedy to this problem is smoothing the field X into a random function X_t, where t indicates the scale at which this smoothing or cutting off happens, defining the measure $\mu_{t,\beta}(dx) = e^{\beta X_t(x)}dx$ and then taking the limit of these measures as the scale at which the smoothing occurs is taken to zero.

As we will see in later chapters, there are still problems in this approach - a t-dependent normalization is required so that the measures converge to a non-trivial limit and this normalization will depend rather delicately on the parameter β. In particular, there is a point β_c, where the behavior of the normalization changes drastically - in physics language, a phase transition occurs.

Instead of going into further details, we will first discuss some problems arising in mathematics and physics where such measures are relevant.

1. Disordered systems - the Random Energy Model

Disordered systems (that is physical models where for example impurities or some other type of disorder is present) still pose challenges even for theoretical physics. Thus studying simple disordered systems rigorously can be interesting: one may hope to find phenomena that are universal and present in more complicated systems. One possible way to model disorder is to introduce a random potential energy coupled to the Hamiltonian of the system. This potential energy models for example how all of the impurities present affect a particle located at the point x in the lattice the system lives on. The randomness models our uncertainty and lack of knowledge of the precise properties of the disorder. One could then hope to classify the effect of the disorder based on its strength (the variance of this random field) and the strength of its correlations (the covariance of the random field). Note that in the type of modelling we have described, the system can not affect the disorder. This is called quenched disorder. The opposite of this - where the randomness evolves with the system - is called annealed disorder.

One of the simplest disordered systems would then be to ignore the Hamiltonian of the 'original system' and consider only the random potential: consider a single particle moving in this random energy field. To make things more precise, consider a lattice $\{-N,...,N\}^d \subset \mathbb{Z}^d$ where the particle moves and assume that at each point of the lattice x, there is a random potential energy $-X(x)$ (the potential will depend on N, but we will repress this in our notation). Then at an equilibrium configuration, the probability (Gibbs weight) that the particle is located at x is $\mu_{\beta,N}(x) = Z_{\beta,N}^{-1} e^{\beta X(x)}$, where $Z_{\beta,N} = \sum_{x\in\{-N,...,N\}^d} e^{\beta X(x)}$. So if X has logarithmic correlations, $\mu_{\beta,N}$ is in some sense similar to our original measures of interest. Note that in the lattice case a short range singularity is meaningless and logarithmic correlations mean in fact $\mathbb{E}(X(x)X(y)) \sim \log N - \log|x-y|$. The relationship is similar to that between an infinite volume limit and a continuum limit in statistical mechanics and field theory.

Universal thermodynamic quantities can only appear at the thermodynamic limit, where we let $N \to \infty$. From the point of view of statistical mechanics, some natural questions one could ask are: can we say something about $Z_{\beta,N}$ as $N \to \infty$, can we say something about the free energy $\log Z_{\beta,N}$, can we perform some kind of scaling limit so that $\mu_{\beta,N}$ would give rise to some non-trivial measure and is there a phase transition in the model - is there a value of β at which the behavior of the system changes radically?

To see what kind of results one might expect to obtain, consider the simplest case where $\{V(x)\}_{x\in\{-N,...,N\}^d}$ are i.i.d. random variables - say centered Gaussians. When the variance is chosen to be proportional to $\log N$, this is known as the Random Energy Model and was introduced by Derrida [**22**]. For an extensive introduction to the model and known results, see [**17, 67**]. Later on, we shall be interested in branching random walks for which it is natural to parametrize the spatial coordinates in a different manner: consider $\{1,...,2^N\}^d$ instead of $\{-N,...,N\}^d$. Since in the REM we have no correlations, it shouldn't make a big difference which formulation we use. Moreover, for simplicity, we will set $d = 1$. We will index the points in $\{1,...,2^N\}$ by $\sigma \in \Sigma_N := \{0,1\}^N$ (this indexing will become more intuitive when we talk about the branching random walk). We continue to write $Z_{\beta,N} = \sum_{\sigma\in\Sigma_N} e^{\beta X_\sigma}$ (and now X_σ are i.i.d. Gaussians with variance N).

For the REM a lot is known about the free energy, partition function and Gibbs measures (for more information about the following statements and proofs, see [**17**]).

Theorem 1.1. *Let* $\beta_c = \sqrt{2\log 2}$. *Then*

$$\lim_{N\to\infty} \mathbb{E}\left(-\frac{1}{N}\log Z_{\beta,N}\right) = \begin{cases} -\frac{\beta^2}{2} - \log 2, & \text{for } \beta \le \beta_c \\ -\beta_c\beta, & \text{for } \beta > \beta_c \end{cases}. \tag{1.2}$$

Theorem 1.2. **(i)** For $\beta < \sqrt{2\log 2}$, there exists a deterministic sequence $a_{N,\beta}$ such that $a_{N,\beta}\log\frac{Z_{N,\beta}}{\mathbb{E}(Z_{N,\beta})}$ converges in distribution to a non-trivial random variable as $N \to \infty$.

(ii) For $\beta = \sqrt{2\log 2}$, there exist deterministic sequences a_N and b_N such that $a_N Z_{N,\beta} + b_N$ converges in distribution to a non-trivial random variable as $N \to \infty$.

(iii) For $\beta > \sqrt{2\log 2}$, there exists a deterministic sequence $a_{N,\beta}$ such that $a_{N,\beta} Z_{N,\beta}$ converges to a non-trivial random variable as $N \to \infty$.

Theorem 1.3. *Let*

$$r_N(\sigma) = \sum_{i=1}^{N} (2\sigma_i - 1) 2^{-i} \tag{1.3}$$

and

$$\bar{\mu}_{\beta,N}(dx) = \sum_{\sigma \in \Sigma_n} \delta(x - r_N(\sigma)) \mu_{\beta,N}(\sigma). \tag{1.4}$$

(i) For $\beta \leq \beta_c$,

$$\bar{\mu}_{\beta,N}(dx) \to \frac{1}{2} dx, \tag{1.5}$$

weakly almost surely in the space of probability measures on $[-1,1]$.

(ii) Let $\mathcal{R}$ be a Poisson point process on $[-1,1] \times \mathbb{R}$ with intensity $\frac{1}{2} dy \otimes e^{-x} dx$. If (Y_k, X_k) are the atoms of this point process, define on $[-1,1] \times (0,1]$ the point process $\mathcal{W}_\alpha$ with atoms (Y_k, w_k), where

$$w_k = \frac{e^{\alpha X_k}}{\int \mathcal{R}(dy, dx) e^{\alpha x}}. \tag{1.6}$$

Then for $\beta > \beta_c$ and $\alpha = \frac{\beta}{\beta_c}$

$$\bar{\mu}_{\beta,N}(dx) \xrightarrow{d} \bar{\mu}_\beta(dx) = \int_{[-1,1] \times (0,1]} \mathcal{W}_\alpha(dy, dw) w \delta(y - x) \tag{1.7}$$

weakly.

Let us make a few remarks on these theorems. Theorem 1.1 says that something happens in the model at $\beta = \sqrt{2 \log 2}$ - the limit of the free energy density is not an analytic function of the temperature at this point. We note that in fact $\mathbb{E}(-\frac{1}{\beta N} \log Z_{\beta,N})$ becomes independent of the temperature at low enough temperatures. Theorem 1.2 describes the fluctuations of the partition function. Finally, Theorem 1.3 says that if we take a suitable continuum limit of the measure (we map $\{1, ..., 2^N\}$ to $[-1,1]$ according to a binary expansion), then the Gibbs measures converge. In the high-temperature case, we get a continuous measure with no atoms (the Lebesgue measure). In the low-temperature case we get a purely atomic measure. We note that by taking such a continuum limit, we lose the fine structure of the individual weights $\mu_{\beta,N}(\sigma)$ - nearby points in the σ-space are mapped to the same point in the continuum limit. Thus in the continuum limit, we can not address questions about the fine structure properties of the $\mu_{\beta,N}(\sigma)$.

The above remarks have a 'universal flavor' to them. One might hope that they hold even if there is some correlation in the potential. In the case where the potential has asymptotically logarithmic correlations, these types of systems are studied in [**20**] through numerical simulations and non-rigorous renormalization group arguments. Their results are indeed consistent with the picture that such a 'freezing transition', where some quantities become independent of the temperature at low enough temperatures, occurs in the model. Moreover, they argue that there exists a deterministic scale around which the free energy fluctuates (this deterministic scale corresponding to the deterministic normalization of the partition function as in the REM). Finally they observe through numerics that at high temperatures, the Gibbs measure seems to be spread out - consistent with a non-atomic structure - while at low temperatures, the Gibbs measure is supported on only a few configurations which are separated from each other - this being consistent with the picture that the measure is purely atomic.

Let us make a further comment on the low temperature Gibbs measure. We could just as well use the binary expansion to map the discrete Gibbs measure into a measure on $[0,1]$. If we then consider a Poisson point process on $[0,1]\times\mathbb{R}$ with intensity $dy\otimes e^{-x}dx$ and write its atoms as $(Y_k,X_k)_{k=1}^{\infty}$, using the same notation as in the theorem, we would get

$$\bar{\mu}_\beta([0,t])=\frac{\sum_{k=1}^{\infty}e^{\alpha X_k}\mathbf{1}\{Y_k\in[0,t]\}}{\sum_{k=1}^{\infty}e^{\alpha X_k}}. \tag{1.8}$$

Since $\sum_k \delta_{(Y_k,X_k)}$ is a Poisson point process with intensity measure $dy\otimes e^{-x}dx$, $\eta(dy,dx)=\sum_k\delta_{(Y_k,e^{\alpha X_k})}$ is a Poisson point process with intensity $\alpha^{-1}dy\otimes x^{-1-\frac{1}{\alpha}}dx$. So we can write

$$\bar{\mu}_\beta([0,t])\stackrel{d}{=}\frac{\int_0^t\int_0^\infty x\eta(dy,dx)}{\int_0^1\int_0^\infty x\eta(dy,dx)}. \tag{1.9}$$

The process

$$L_\alpha(t)=\int_0^t\int_0^\infty x\eta(dy,dx) \tag{1.10}$$

is a well known stochastic process: it is a stable Lévy subordinator of index $\frac{1}{\alpha}$ (see e.g. [**45**]). It is a non-negative pure jump process. In conclusion, we can write the limiting Gibbs measure in terms of stable Lévy subordinator of index $\frac{1}{\alpha}$:

$$\bar{\mu}_\beta([0,t])\stackrel{d}{=}\frac{L_\alpha(t)}{L_\alpha(1)}. \tag{1.11}$$

We shall see later on that for a special family of correlated random variables (namely the branching random walk), the corresponding low temperature measure has been shown to exist and it has a representation (with corresponding notation)

$$\mu_\beta([0,t])\stackrel{d}{=}\frac{L_\alpha(\mu_{\beta_c}([0,t]))}{L_\alpha(\mu_{\beta_c}([0,1]))} \tag{1.12}$$

suggesting that the Lévy process perhaps plays some universal role in the low temperature measures.

We finish this section with a brief remark about the limit of the expectation of the free energy in the REM. As we noted, $\lim_{N\to\infty}\mathbb{E}(-\frac{1}{\beta N}\log Z_{\beta,N})=-\frac{\beta}{2}-\frac{\log 2}{\beta}$ for $\beta\leq\beta_c$ and is constant for $\beta\geq\beta_c$. Note that the function $\beta\to\frac{\beta}{2}+\frac{\log 2}{\beta}$ is invariant under the transformation $\beta\to\frac{\beta_c^2}{\beta}$ - i.e. inversion with respect to the critical point. A similar phenomenon was noted in [**33**]. They consider a model with logarithmic correlations and observe this type of duality in an observable that freezes at the critical point. This lead them to make a conjecture that this type of formal duality is a sign of an observable freezing at the critical point. Further evidence for this was found in [**34**].

2. A massless two-dimensional Dirac fermion in a random magnetic field

As an example of a different type of physical model where such a measure appears, consider a massless two-dimensional Dirac fermion in an external magnetic field perpendicular to the plane the particle is in. Low-dimensional electronic systems in strong magnetic fields are known to be a source of many interesting physical effects with potential technological applications (such as the quantum Hall effect and its applications to topological quantum computing [**57**]). Thus our example is not completely artificial. Our presentation follows [**20, 56**] and is not mathematically rigorous, but on the level of theoretical physics. Consider the Hamiltonian

$$H = \sigma_\mu (i\partial_\mu - A_\mu(x)), \tag{1.13}$$

where $(\sigma_\mu)_{\mu=1,2}$ are the first two Pauli spin-matrices and A_μ is the vector potential for the magnetic field we are considering and we have chosen units so that the characteristic velocity of particles in the model is one. We also use the Einstein summation convention, i.e. we sum over repeated indexes.

Let us work in the Coulomb gauge ($\partial_\mu A_\mu = 0$). In this case, we can find a 'scalar potential' for the vector potential: a function ϕ so that $A_\mu = \epsilon_{\mu\nu}\partial_\nu \phi$, where $\epsilon_{\mu\nu}$ is the totally antisymmetric tensor. This yields $B = -\Delta\phi$.

We consider now the situation where the Gauge potential A_μ is our fundamental random variable. We take (A_1, A_2) to be a centered Gaussian vector with short range correlations: $\mathbb{E}(A_\mu(x)A_\nu(y)) \sim \delta_{\mu,\nu}\delta(x-y)$. We thus give a configuration the weight (assuming suitable decay at infinity)

$$Ce^{-\alpha\int_{\mathbb{R}^2} A_\mu(x)\cdot A_\mu(x)dx} = Ce^{-\alpha\int_{\mathbb{R}^2}(\partial_\mu\phi(x))^2 dx} = Ce^{-\alpha\int_{\mathbb{R}^2}\phi(-\Delta\phi)dx}. \tag{1.14}$$

This is simply the weight in the path integral representation of a massless Bosonic field in two-dimensions so if we regularize our model by putting it on a lattice of lattice constant a, the correlations are given (for $a << |x-y| << L$, where L is the size of our system that plays no real role for us) by

$$\mathbb{E}\left((\phi(x)-\phi(y))^2\right) \sim \log\frac{|x-y|}{a}. \tag{1.15}$$

Consider then a zero energy (ground) state $\Psi = (\Psi_1, \Psi_2)$ of the system: writing the Hamiltonian and matrices out explicitly, we see that $H\Psi = 0$ is equivalent to

$$\begin{pmatrix} (i\partial_1+\partial_2)\Psi_2 - ((i\partial_1+\partial_2)\phi)\Psi_2 \\ (i\partial_1-\partial_2)\Psi_1 + ((i\partial_1-\partial_2)\phi)\Psi_1 \end{pmatrix} = 0. \tag{1.16}$$

The two components are independent so we can find two independent solutions $\Psi = (\psi, 0)$ and $\Psi = (0, \tilde{\psi})$, where $\psi(x)$ is proportional to $e^{\phi(x)}$ and $\tilde{\psi}(x)$ is proportional to $e^{-\phi(x)}$. If we take Ψ to be normalized, we find

$$\psi(x)^2 = \frac{e^{2\phi(x)}}{\int_{\mathbb{R}^2} e^{2\phi(x)}dx} \tag{1.17}$$

and in $\tilde{\psi}(x)^2$ we replace ϕ by $-\phi$. So we see that the ground state probability density is a normalized measure of the form $Z_\beta^{-1}e^{\beta X(x)}dx$, where X is a Gaussian field with logarithmic correlations and β is proportional to $\alpha^{-\frac{1}{2}}$.

3. Extreme value statistics

Consider a situation where we have done the regularization of the field X by setting it to be constant on dyadic intervals. Let us index dyadic intervals of length 2^{-n} by $\sigma \in \Sigma_n = \{0,1\}^n$. The identification is that 0 means 'choosing left' and 1 means 'choosing right'. For example, we identify (0) with the interval $[0, \frac{1}{2}]$, (1) with the interval $[\frac{1}{2}, 1]$ and $(0,1)$ with the interval $[\frac{1}{4}, \frac{1}{2}]$.

Consider then the mass of the unit interval for the approximating measure at level n (the mass of the interval identified with σ is $\int_\sigma e^{\beta X_\sigma}dx = e^{\beta X_\sigma}2^{-n}$):

$$\mu_{n,\beta}([0,1]) = 2^{-n}\sum_{\sigma\in\Sigma_n} e^{\beta X_\sigma} \tag{1.18}$$

and the generating function

$$G_{n,\beta}(x) = \mathbb{E}(\exp(-e^{-\beta(x-\frac{n\log 2}{\beta})}\mu_{n,\beta}([0,1]))). \tag{1.19}$$

We then note that almost surely

$$\lim_{\beta\to\infty}\exp\left(-\sum_{\sigma\in\Sigma_n} e^{\beta(X_\sigma - x)}\right) = \mathbf{1}\left\{\max_{\sigma\in\Sigma_n} X_\sigma < x\right\}, \tag{1.20}$$

so

$$G_{n,\infty}(x) = \mathbb{P}\left(\max_{\sigma\in\Sigma_n} X_\sigma < x\right). \tag{1.21}$$

and knowledge about $\mu_{n,\infty}$ translates into knowledge about the distribution of $\max_{\sigma\in\Sigma_n} X_\sigma$.

This is a rather non-trivial mathematical problem in the case where the $(X_\sigma)_{\sigma\in\Sigma_n}$ are correlated. For example, it is known that for weak enough correlations, a large class of families belong to the Gumbel universality class: there exist deterministic a_n and b_n such that $\mathbb{P}(\max_{\sigma\in\Sigma_n} X_\sigma \le a_n x + b_n) \to \exp(-\exp(-x))$ (see for example [**49**] and the appendix of [**20**]). Recently, several examples have been found of logarithmically correlated fields where we are not in the Gumbel case (see [**19, 24, 32, 68**]). Indeed, in [**32**] the full probability distribution was found (through non-rigorous methods based on the assumption of a freezing transition) in a specific model of logarithmically correlated random variables and this is not the Gumbel law. Moreover, in [**20**] it is conjectured that for logarithmically correlated models, the tail of the distribution of the maximum is universal. In the examples mentioned, it is seen to be of the form $xe^{-\alpha x}$, where α is a constant related to the variance of the field.

4. Random geometry, Quantum gravity and KPZ

Perhaps the application of such measures that has attracted the most interest in the past few years comes from string theory. It can be also formulated as a problem of statistical mechanics of random surfaces (the surface playing the role of the world sheet of the string). We will follow [**59**] to describe the physical model. Consider a model of statistical mechanics where we embed a two-dimensional surface in d-dimensional space and assume that the surface has an internal degree of freedom one can interpret as a metric. The dynamical variables of the model are then the coordinates of the embedding $(X^\mu(\tau,\sigma))_{\mu=1}^d \in \mathbb{R}^d$ (τ and σ parametrize the surface) and the metric of the surface $g(\tau,\sigma)$. We then wish to give a configuration (X^μ, g) the weight $e^{-S(X^\mu,g)}$, where S is an action functional which is both relatively simple and satisfies the symmetries we would like our model to have.

Let us now motivate our choice of S through some symmetry arguments. First of all, we do not prefer any specific location or direction in space so it makes sense to demand that S is invariant under global rotations and global translations of the X^μ-coordinates. Moreover, we consider only the surface to be significant - not its parametrization so we want invariance under general coordinate transformations of the (τ,σ) coordinates - i.e. diffeomorphism invariance in the (τ,σ)-space.

Perhaps the simplest choice of an action that satisfies both of these conditions is

$$S(X^\mu, g) = \alpha \int d\tau d\sigma \sqrt{\det(g)}(g_{a,b}\partial_a X^\mu \partial_b X^\mu + \lambda), \tag{1.22}$$

where λ is some constant (cosmological constant). We also sum over repeated indexes, a and b run over $1,2$ (coordinates in the (τ,σ)-space) and μ runs over $1,...,d$. That the action is invariant under translations of X^μ is clear since we take the derivative of the X^μ-coordinates. Invariance under rotations is also clear since we are summing over μ, i.e. taking an inner product of $\partial_a X^\mu$ and $\partial_b X^\mu$ in $\mathbb{R}^d$ and rotations leave inner products invariant. To see invariance under diffeomorphisms in the (τ,σ)-space, let us write $x=(\tau,\sigma)$ and assume that $x \mapsto \tilde{x}(\tau,\sigma)$ is a diffeomorphism. Then g transforms according to $g \mapsto \tilde{g}$ with $\tilde{g}_{ab} = g_{cd}\frac{\partial \tilde{x}_a}{\partial x_c}\frac{\partial \tilde{x}_b}{\partial x_d}$ so

$$g_{ab}\partial_a X^\mu \partial_b X^\mu \mapsto g_{cd}\frac{\partial \tilde{x}_a}{\partial x_c}\frac{\partial \tilde{x}_b}{\partial x_d}\frac{\partial}{\partial \tilde{x}_a}X^\mu \frac{\partial}{\partial \tilde{x}_b}X^\mu = g_{cd}\partial_c X^\mu \partial_d X^\mu. \tag{1.23}$$

Moreover, if J is the Jacobian of the coordinate transformation, then

$$d\tau d\sigma\sqrt{\det(g)} \mapsto J d\tau d\sigma \sqrt{\det(\tilde{g})} = J d\tau d\sigma \sqrt{J^{-2}\det(g)} = d\tau d\sigma \sqrt{\det(g)} \tag{1.24}$$

and we see that the action is indeed invariant under such transformations.

Using the diffeomorphism invariance of the action, we can reduce the three degrees of freedom of the metric tensor to a single scale factor: $\hat{g}_{ab} = e^{2\phi}\delta_{ab}$ (the conformal gauge).

In the path integral, we can try to mod out the over counting from diffeomorphism invariance using the Faddeev-Popov trick:

$$\int DX^\mu Dg e^{-S(X^\mu,g)} = \int DX^\mu D\phi e^{-S(X^\mu,\hat{g})}\Delta_{FP}(\hat{g}). \tag{1.25}$$

Referring to [**59**] for the details of calculating the Faddeev-Popov determinant $\Delta_{FP}(\hat{g})$, we see that the Faddeev-Popov procedure leaves us with an effective action

$$S(X^\mu, \delta e^{2\phi}) = \alpha \int d\tau d\sigma \left(\partial_a X^\mu \partial_a X^\mu + \lambda e^{2\phi} + C\partial_a\phi\partial_a\phi\right). \tag{1.26}$$

If we consider the embedding of the surface to be fixed, i.e. just discarding X^μ from the problem, this is known as the action of Liouville gravity - a model of quantum gravity in 1+1-dimensional space-time (although this is in flat space - in general it is in curved space). For a fixed embedding, the only dynamical variable in the model is the field ϕ. So we have a metric $e^{2\phi}\delta$ and the area measure will be $e^{2\phi}d\sigma d\tau$. In the case where $\lambda = 0$ (the so called critical string) the action reduces to that of a free massless boson. So when $\lambda = 0$, we get an area measure which is precisely of the form $e^{\beta X(x)}dx$, where X has logarithmic correlations.

This non-rigorous argument has inspired people to study measures of the form $e^{\beta X(x)}dx$ rigorously and hope to be able to give precise meaning to ideas and results from two-dimensional quantum gravity. A very curious result from quantum gravity is known as the KPZ-relation [**47, 21**]. The non-rigorous 'physics-version' of the KPZ-relation is an algebraic relation between the conformal weights Δ^0 of the primary operators of a two-dimensional conformal field theory and the scaling dimensions Δ of these fields when the conformal field theory is coupled to two-dimensional quantum gravity (so Δ and Δ_0 are essentially the algebraic decay rates at infinity of the two point function in the two different theories). The relation is

$$\Delta^0 = \Delta + \frac{\gamma^2}{4}\Delta(\Delta - 1), \tag{1.27}$$

where $\gamma = \sqrt{(25-c)/6} - \sqrt{(1-c)/6}$ and c is the central charge of the conformal field theory.

One possible application to this formula comes from the fact that the conformal field theory appearing in it could be taken to be a scaling limit of some model of statistical mechanics on a two-dimensional lattice. While the model where the conformal field theory is coupled to gravity can be understood as a model of statistical mechanics on a random lattice. In some cases it could occur (see e.g. [**26**]) that calculating the quantity Δ on the random lattice would be much easier that calculating Δ^0 so this relation would give a way of calculating quantities of two-dimensional models of statistical mechanics. Even more rigorously, there are lattice models on random lattices expected to correspond to a model of statistical physics coupled to quantum gravity where some quantities can be calculated through random matrix theory [**40**].

While currently it seems that making mathematically rigorous sense about this relation and the program of relating statistical mechanics models on random and non-random lattices is still far away, there has been recently rigorous results on some geometric quantities which obey the same functional KPZ-relation though their interpretation is different. More precisely, the following theorem was proven in [**29**] (see also [**36**] for further information about the mathematical KPZ-relation and conjectures about its relation to statistical mechanics).

THEOREM 1.4. *Let X be the Gaussian free field in a simply connected bounded planar domain D with zero boundary conditions, $\beta \in [0,2)$ and $\mu_\beta(dx) = e^{\beta X(x)}dx$ (where the proper definition requires a smoothing, normalization and limiting procedure) and $\tilde{D}$ a deterministic subset which lies within a compact subset of D.*

Let $\delta > 0$, $B_r(z)$ be the Euclidean ball of radius r around z, $\epsilon(z,\delta) = \sup\{r > 0 : \ \mu_\beta(B_r(z)) \le \delta\}$ and let $B^\delta(z) = B_\epsilon(z)$ (the ball around z with 'quantum area' δ). One then defines the Euclidean scaling exponent (we write μ_0 for the Lebesgue measure on $\mathbb{R}^2$)

$$\Delta^0(\tilde{D}) = \lim_{\epsilon \to 0} \frac{\log \mu_0\left(\{z : \ B_\epsilon(z) \cap \tilde{D} \neq \emptyset\}\right)}{\log \epsilon^2} \tag{1.28}$$

and quantum scaling exponent

$$\Delta(\tilde{D}) = \lim_{\delta \to 0} \frac{\log \mathbb{E}\left(\mu_\beta\left(\{z : \ B^\delta(z) \cap \tilde{D} \neq \emptyset\}\right)\right)}{\log \delta} \tag{1.29}$$

and if $\Delta^0(\tilde{D}) \ge 0$, then $\Delta(\tilde{D})$ satisfies the equation

$$\Delta^0(\tilde{D}) = \Delta(\tilde{D}) + \frac{\beta^2}{4}\Delta(\tilde{D})(\Delta(\tilde{D}) - 1). \tag{1.30}$$

Inspired by this result, there has been much research into similar results for other measures, results up to the critical point $\beta = 2$ and results concerning a KPZ-relation after the critical point: $\beta > 2$ ([**13, 9, 10, 60, 28**]). The identical form of the algebraic relations suggests that there should be something further to understand between 'quantum gravity' and random geometry related to measures of the form $e^{\beta X(x)}dx$, but a lot is still not understood. For example, a proper understanding of the metric $e^{2\phi}\delta$ is still lacking, though it is conjectured that 'the Brownian map' ([**51, 54**])- an object constructed as a limit of random planar maps - could be interpreted as the random metric space relevant to quantum gravity. In [**38, 14**] another approach is taken to make sense of quantum gravity and further understand such random geometry through constructing the natural Brownian motion in such geometry. This work also allows giving rigorous meaning to some path integrals appearing in quantum gravity.

5. Conformal Welding

Random (conformally invariant) planar curves have been of great interest since the discovery of SLE [**63**]. In [**6**] a method of of constructing random closed planar curves through conformal welding is discussed. Conformal welding is the procedure of constructing a planar Jordan curve from a suitable homeomorphism of the circle $\mathbb{S}^1$. More precisely, one is given a homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ and one is asked to find a planar Jordan curve Γ such that if $f_+ : \mathbb{D} \to \Omega_+$ and $f_- : \mathbb{D}_\infty \to \Omega_-$ are the Riemann mappings of the unit disc $\mathbb{D}$ and the exterior of the unit disc $\mathbb{D}_\infty$ onto the components of the exterior of Γ: $\hat{\mathbb{C}} \setminus \Gamma = \Omega_+ \cup \Omega_-$, then $h = (f_+)^{-1} \circ f_-$ ($f_\pm$ can be extended continuously to $\mathbb{S}^1$ by Caratheodory's theorem).

One method of solving the welding problem is through the Beltrami equation in a domain $\Omega \subset \mathbb{C}$:

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z} \tag{1.31}$$

for almost every $z \in \Omega$, where one looks for homeomorphic solutions f (with certain further regularity). If one assumes $||\mu||_\infty < 1$, the solutions are unique up to conformal mappings. Let us briefly see how this uniqueness of the Beltrami equation is related to the welding problem in the case $||\mu||_\infty < 1$. Assume that f is a solution to the Beltrami equation in the disc $\mathbb{D}$ for some μ with $||\mu||_\infty < 1$ and $f|_{\mathbb{S}^1} = h$ for the given circle homeomorphism. Consider then the auxiliary equation

$$\frac{\partial F}{\partial \bar{z}} = \mathbf{1}\{z \in \mathbb{D}\}\mu(z)\frac{\partial F}{\partial z}, \tag{1.32}$$

for almost every $z \in \mathbb{C}$. Then $\Gamma = F(\mathbb{S}^1)$ is a Jordan curve and since F is conformal in $\mathbb{D}_\infty$, $f_- = F|_{\mathbb{D}_\infty} : \mathbb{D}_\infty \to \Omega_- = F(\mathbb{D}_\infty)$ defines a conformal mapping. On the other hand, both f and F solve the Beltrami equation in the disc so by uniqueness, $F = f_+ \circ f$ for some conformal $f_+ : \mathbb{D} \to \Omega_+ = F(\mathbb{D})$. Thus on $\mathbb{S}^1$, $\phi = f|_{\mathbb{S}^1} = (f_+)^{-1} \circ f_-$ and we have a solution to the welding problem.

To construct random planar curves through welding, one considers random circle homeomorphisms. The random homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ that is considered in [**6**] is based on the centered Gaussian field X on $\mathbb{S}^1$ with covariance $\mathbb{E}(X(z)X(z')) = -\log|z - z'|$. They define the measure τ on $\mathbb{S}^1$ by

$$\tau(dz) = \lim_{\epsilon \to 0} \frac{e^{\beta X_\epsilon(z)}}{\mathbb{E}e^{\beta X_\epsilon(z)}}dz, \tag{1.33}$$

for $\beta < \sqrt{2}$ ($\beta_c = \sqrt{2}$ turns out to be the 'critical point' in this model). Here X_ϵ is a suitable regularization of the field X at the scale ϵ. Identifying $\mathbb{S}^1$ with $[0, 1)$, they define the homeomorphism $h : [0, 1) \to [0, 1)$ by

$$h(\theta) = \frac{\tau([0, \theta))}{\tau([0, 1))}. \tag{1.34}$$

The difficulty in this case is that the μ appearing in the (random) Beltrami equation related to this h will not satisfy the $||\mu||_\infty < 1$ condition and more elaborate arguments are required.

6. Number Theory

We end this motivational chapter by discussing an interesting relationship between the Riemann ζ-function and logarithmically correlated Gaussian fields. The starting point for this relationship is the following central limit theorem discovered by Selberg [**64, 65**]:

$$\lim_{T \to \infty} \frac{1}{T}\lambda\left(\left\{t \in [T, 2T] : \ \alpha \le \frac{\log|\zeta(\frac{1}{2} + it)|}{\sqrt{\frac{1}{2}\log\log\frac{t}{2\pi}}} \le \beta\right\}\right) = \int_\alpha^\beta \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx, \tag{1.35}$$

i.e. that in a sense, $\log|\zeta(\frac{1}{2}+it)|$ behaves like a Gaussian random variable as $t \to \infty$. Moreover, it was noted in [**35**] that the correlations of this random variable are logarithmic: define $V_t(x) = -2\log|\zeta(\frac{1}{2}+i(t+x))|$. Then for $\frac{1}{\log t} << h << t$ and as $t \to \infty$

(1.36)
$$\frac{1}{h}\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} V_t(x_1+y)V_t(x_2+y)dy \approx \begin{cases} -\log|x_1-x_2|, & \frac{1}{\log t} << |x_1-x_2| << t \\ 2\log\log t, & |x_1-x_2| << \frac{1}{\log t} \end{cases}.$$

In [**35**] it is further conjectured through relationships to random matrices and disordered systems that the freezing transition scenario might be relevant to studying extreme values of the Riemann ζ-function.

CHAPTER 2

Logarithmically correlated fields

In this section, we consider several different fields which have a short range logarithmic singularity in their covariances. We first discuss the two-dimensional Gaussian Free Field which is the mathematically rigorous definition of a free massless bosonic field in two dimensions (in imaginary time). We then discuss the $1/f$ or pink noise which is a one-dimensional Gaussian process whose power spectral density is inversely proportional to its frequency. It occurs in many physical, biological end economical models. We also consider some Gaussian fields with logarithmic correlations through a white noise decomposition. We also discuss branching random walks which can be interpreted as an approximation or simplification to the logarithmically correlated fields. We mention a continuum version of branching random walks - namely branching Brownian motion. Finally we discuss some tools to analyze functionals of these fields - in particular we recall some results from renewal theory.

1. The Gaussian Free Field

The Gaussian Free Field (GFF) on a set (usually a domain in $\mathbb{R}^d$ or a graph) is simply a Gaussian field whose covariance is given by the inverse of the Laplace operator in this set. For the existence and uniqueness of this inverse, one needs to specify boundary conditions for the Laplacian. We shall here give two examples of the GFF: one discrete and one in the continuum.

For simplicity, let us consider the two-dimensional discrete free field with Dirichlet boundary conditions on $S_N = (N^{-1}\mathbb{Z})^2 \cap [0,1]^2$. We shall also study its covariance. Our discussion is similar to [**39**]. For a vector $\phi = (\phi_x)_{x\in(N^{-1}\mathbb{Z})^2}$, define for $x \in (N^{-1}\mathbb{Z})^2$

$$(-\Delta\phi)_x = N^2 \sum_{y\in(N^{-1}\mathbb{Z})^2:\ |x-y|=N^{-1}} (\phi_x - \phi_y). \tag{2.1}$$

We impose Dirichlet boundary conditions by taking $\phi_y = 0$ when $y \notin S_N^{\mathrm{int}} = (N^{-1}\mathbb{Z})^2 \cap (0,1)^2$. We then define the probability measure on $\mathbb{R}^{S_N^{\mathrm{int}}}$:

$$\mathbb{P}(d\phi) = \frac{1}{Z} e^{-\frac{1}{2N^2}\sum_{x\in S_N^{\mathrm{int}}} \phi_x(-\Delta\phi)_x} \prod_{x\in S_N^{\mathrm{int}}} d\phi_x, \tag{2.2}$$

where $d\phi_x$ is the Lebesgue measure on $\mathbb{R}$ and Z is a normalizing constant. We note Dirichlet boundary conditions imply that

(2.3)
$$\sum_{x\in S_N^{\mathrm{int}}}\sum_{\substack{y\in S_N:\\|x-y|=N^{-1}}}\phi_x(\phi_x-\phi_y)=\sum_{\substack{x,y\in S_N:\\|x-y|=N^{-1}}}(\phi_x-\phi_y)^2-\sum_{\substack{x,y\in S_N:\\|x-y|=N^{-1}}}\phi_y(\phi_y-\phi_x)$$
$$=\frac{1}{2}\sum_{\substack{x,y\in S_N:\\|x-y|=N^{-1}}}(\phi_x-\phi_y)^2.$$

From this representation, we see that the discrete GFF is in a sense a natural generalization of a random walk into a process with a two-dimensional time parameter: the increments $\phi_x-\phi_y$ for x,y nearest neighbors are centered Gaussians.

To see that the Gaussian density is in fact integrable and to analyze the covariance of this field, we note that Δ can be diagonalized by going into Fourier modes: define $e^k(x)=2\sin(k_1x_1)\sin(k_2x_2)$ for $k=(k_1,k_2)\in\{\pi,2\pi,...,(N-1)\pi\}^2$ and $x=(x_1,x_2)\in S_N$. One can then check that on S_N

$$-\Delta e^k=4N^2\left(\sin^2\frac{k_1}{2N}+\sin^2\frac{k_2}{2N}\right)e^k. \tag{2.4}$$

Moreover, $(e^k)_k$ is an orthonormal basis (with respect to the standard ℓ_2 inner product) for functions $\phi\in\mathbb{R}^{S_N}$ which vanish outside of S_N^{int}. Thus Δ is positive definite, $\mathbb{P}(d\phi)$ is well defined and for $x,y\in S_N^{\mathrm{int}}$

$$(-\Delta)^{-1}_{x,y}=\sum_{1\leq\frac{k_1}{\pi},\frac{k_2}{\pi}\leq N-1}\frac{1}{4N^2\left(\sin^2\frac{k_1}{2N}+\sin^2\frac{k_2}{2N}\right)}e_x^{(k_1,k_2)}e_y^{(k_1,k_2)}. \tag{2.5}$$

Another way to describe this covariance is through Green's functions: let $(s_n)_n$ be a simple random walk in $(N^{-1}\mathbb{Z})^2$ and let τ be the time at which s_n first hits $\partial[0,1]^2$. For $x,y\in S_N^{\mathrm{int}}$, define

$$G(x,y)=\frac{1}{4N^2}\mathbb{E}^x\left(\sum_{n=0}^{\tau-1}\mathbf{1}\{s_n=y\}\right), \tag{2.6}$$

i.e. $G(x,y)$ is the expected number of visits to y before exiting S_N^{int} when we start the walk from x.

To show that $G=-\Delta^{-1}$, note that for a function $f\in\mathbb{R}^{S_N}$ vanishing on $\partial[0,1]^2$ and for a point $x\in S_N^{\mathrm{int}}$, $(-\Delta f)(x)=-4N^2(\mathbb{E}^x(f(s_1))-f(x))$, where s_n is a simple random walk on $(N^{-1}\mathbb{Z})^2$ so for $x,y\in S_N^{\mathrm{int}}$

$$
\begin{aligned}
-\Delta_x G(x,y) &= -4N^2(\mathbb{E}^x(G(s_1,y)) - G(x,y)) \\
&= -\mathbb{E}^x\left(\mathbb{E}^{s_1}\left(\sum_{n=0}^{\tau-1}\mathbf{1}\{\tilde{s}_n = y\}\right)\right) + \mathbb{E}^x\left(\sum_{n=0}^{\tau-1}\mathbf{1}\{s_n = y\}\right) \\
&= -\mathbb{E}^x\left(\sum_{n=1}^{\tau}\mathbf{1}\{s_n = y\}\right) + \mathbb{E}^x\left(\sum_{n=0}^{\tau-1}\mathbf{1}\{s_n = y\}\right) \\
&= \mathbb{E}^x(\mathbf{1}\{s_0 = y\}) - \mathbb{E}^x(\mathbf{1}\{s_\tau = y\}) \\
&= \delta_{x,y},
\end{aligned} \tag{2.7}
$$

where $\tilde{s}$ is an independent copy of s.

Estimates of the covariance can then either be done analytically or through random walk theory. Let us consider for example the variance at the point $(\frac{1}{2}, \frac{1}{2})$ as $N \to \infty$. As $N \to \infty$, we note that the eigenvalue of the Laplacian can be estimated by $k_1^2 + k_2^2$ (their ratio is bounded) so

$$
\begin{aligned}
(-\Delta_{(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2})})^{-1} &\sim \sum_{1\leq n_1,n_2\leq N-1}\frac{1}{n_1^2+n_2^2}\sin^2\frac{n_1\pi}{2}\sin^2\frac{n_2\pi}{2} \\
&\sim \sum_{1\leq n_1,n_2\leq N-1}\frac{1}{n_1^2+n_2^2} \\
&\sim \log N.
\end{aligned} \tag{2.8}
$$

One can show that this holds in the 'bulk', i.e. for points macroscopically far away from the boundary. For the covariance, let us use a random walk estimate. Let τ_y be the hitting time of the point $y \in (N^{-1}\mathbb{Z})^2$ so we see by the (strong) Markov property of the random walk that for $x, y \in S_N^{\mathrm{int}}$

$$
\begin{aligned}
G(x,y) &= \frac{1}{4N^2}\mathbb{E}^x\left(\sum_{n=0}^{\tau-1}\mathbf{1}\{s_n = y\}\right) \\
&= \frac{1}{4N^2}\mathbb{E}^x\left(\mathbf{1}\{\tau_y < \tau\}\sum_{n=0}^{\tau_y-1}\mathbf{1}\{s_n = y\} + \mathbf{1}\{\tau_y < \tau\}\sum_{n=\tau_y}^{\tau-1}\mathbf{1}\{s_n = y\}\right) \\
&= \frac{1}{4N^2}\mathbb{P}^x(\tau_y < \tau)\mathbb{E}^y\left(\sum_{n=0}^{\tau-1}\mathbf{1}\{s_n = y\}\right) \\
&= \mathbb{P}^x(\tau_y < \tau)G(y,y).
\end{aligned} \tag{2.9}
$$

To estimate the probability, we first note that by the translation invariance of the simple random walk, our problem is identical to calculating $\mathbb{P}^x(\tau_0 < \tau_{\partial[-\frac{1}{2},\frac{1}{2}]^2})$ ($\tau_{\partial[-\frac{1}{2},\frac{1}{2}]^2}$ is the hitting time of $\partial[-\frac{1}{2},\frac{1}{2}]^2$) for the simple random walk on $(N^{-1}\mathbb{Z})^2$. We then introduce (s is still the simple random walk on $(N^{-1}\mathbb{Z})^2$)

$$
a(x) = \sum_{n=0}^{\infty}(\mathbb{P}^0(s_n = 0) - \mathbb{P}^0(s_n = x)) \tag{2.10}
$$

and the bounded martingale $M_j = a(S_{j\wedge\tau_0\wedge\tau_{\partial[-\frac{1}{2},\frac{1}{2}]^2}})$ (boundedness follows for example from the estimate $|\mathbb{P}^0(s_n = 0) - \mathbb{P}^0(s_n = x)| \leq Cn^{-\frac{3}{2}}N|x|$). By optional stopping and noting that $a(0) = 0$,

$$\begin{aligned} a(x) &= \mathbb{E}^x\left(M_{\tau_0\wedge\tau_{\partial[-\frac{1}{2},\frac{1}{2}]^2}}\right) \\ &= (1-\mathbb{P}^x(\tau_0 < \tau_{\partial[-\frac{1}{2},\frac{1}{2}]^2}))\mathbb{E}^x\left(a\left(S_{\tau_0\wedge\tau_{\partial[-\frac{1}{2},\frac{1}{2}]^2}}\right)\middle| S_{\tau\wedge\tau_y} \in \partial\left[-\frac{1}{2},\frac{1}{2}\right]^2\right). \end{aligned} \tag{2.11}$$

Next we make use of the estimate $a(x) = C_1 \log|Nx| + C_2 + \mathcal{O}((|Nx|)^{-2})$ as $|Nx| \to \infty$. For details, see [**50**]. Thus we conclude that for x and y in the bulk

$$\mathbb{P}^x(\tau_y < \tau) \sim -\frac{\log|x-y|}{\log N} \tag{2.12}$$

and $G(x,y) \sim -\log|x-y|$ for $x-y$ in the bulk.

Finally let us consider an explicit construction for the free field. Consider a sequence of i.i.d. Gaussians $(a_{(n_1,n_2)})_{n_1,n_2=1}^{\infty}$ and for $x \in S_N$

$$\phi(x) = \sum_{n_1,n_2=1}^{N-1} a_{(n_1,n_2)} \frac{1}{2\sqrt{N^2\left(\sin^2\frac{\pi n_1}{2N} + \sin^2\frac{\pi n_2}{2N}\right)}} \sin(n_1\pi x_1)\sin(n_2\pi x_2). \tag{2.13}$$

Then clearly $\phi(x)$ is a centered Gaussian process on S_N with Dirichlet boundary conditions and

$$\mathbb{E}(\phi(x)\phi(y)) = (-\Delta^{-1})_{x,y}, \tag{2.14}$$

so ϕ is indeed the discrete Gaussian free field on S_N with Dirichlet boundary conditions.

The natural way we would like to define a continuum GFF would be to take an $N \to \infty$ limit of the discrete version. The main problem in this is that the limit would have infinite variance: $\mathbb{E}(\phi_x^2) \sim \log N$ in the bulk. It turns out that the proper way to interpret the limiting object is as a random distribution ([**66, 41, 25**]).

Again for simplicity, let us stick to the domain $[0,1]^2$ and let $(a_{(n_1,n_2)})_{n_1,n_2=1}^{\infty}$ be a sequence of i.i.d. Gaussians. Let us consider at first only formally the $N \to \infty$ limit of our representation of the discrete GFF in terms of the i.i.d. Gaussians: for $x \in [0,1]^2$, let

$$\phi(x) = \sum_{n_1,n_2=1}^{\infty} \frac{2}{\pi}\frac{1}{\sqrt{n_1^2+n_2^2}} a_{(n_1,n_2)} \sin(n_1\pi x_1)\sin(n_2\pi x_2). \tag{2.15}$$

The question is now to find in which space such a series would converge almost surely. Consider the Sobolev space $\mathcal{H}_0^1((0,1)^2)$ that is the Hilbert space closure of the set of L^2 functions on $(0,1)^2$ with compact support in $(0,1)^2$ whose derivative is also in L^2. The inner product is

$$(f,g)_{\mathcal{H}_0^1} = (f,g)_{L^2} + (\partial_1 f, \partial_1 g)_{L^2} + (\partial_2 f, \partial_2 g)_{L^2}. \tag{2.16}$$

An orthonormal basis for this space is given by normalizing the sequence $(\sin(n_1\pi x_1)\sin(n_2\pi x_2))_{n_1,n_2=1}^{\infty}$. Let us stick to the non-normalized basis and let $\alpha_{(n_1,n_2)}$ be the coefficients of f in this basis expansion. The condition that f has finite $\mathcal{H}_0^1$ norm can then be expressed as $\sum_{n_1,n_2=1}^{\infty} |\alpha_{(n_1,n_2)}|^2(n_1^2+n_2^2) < \infty$. Consider then the following space of sequences

$$\mathcal{H}_0^{-1}((0,1)^2) = \left\{ (\gamma_{(n_1,n_2)})_{n_1,n_2=1}^{\infty} : \sum_{n_1,n_2=1}^{\infty} (n_1^2+n_2^2)^{-1}|\gamma_{(n_1,n_2)}|^2 < \infty \right\}.$$

In fact, one can check that this is a Banach space with the following norm: for $\psi = (\gamma_{(n_1,n_2)})$, define

$$||\psi||_{\mathcal{H}_0^{-1}}^2 = \sum_{n_1,n_2=1}^{\infty} |\gamma_{(n_1,n_2)}|^2(n_1^2+n_2^2)^{-1}. \tag{2.17}$$

We can then interpret ψ as a distribution acting on $\mathcal{H}_0^1((0,1)^2)$ through the following identification:

$$\psi(f) = \sum_{n_1,n_2=1}^{\infty} \gamma_{(n_1,n_2)}\alpha_{(n_1,n_2)}. \tag{2.18}$$

This series converges since by Cauchy-Schwarz $|\psi(f)| \leq C||\psi||_{\mathcal{H}_0^{-1}}||f||_{\mathcal{H}_0^1}$ for some constant C. Going back to our field, we see that if we identify ϕ with its sequence of coefficients, then

$$||\phi||_{\mathcal{H}_0^{-1}}^2 \sim \sum_{n_1,n_2=1}^{\infty} a_{(n_1,n_2)}^2(n_1^2+n_2^2)^{-2}. \tag{2.19}$$

This series converges almost surely e.g. by Kolmogorov's three series theorem. Thus we can interpret ϕ as a random element in the space of distributions $\mathcal{H}_0^{-1}((0,1)^2)$.

This type of construction of the free field in a more general domain through Sobolev spaces is also possible though such an explicit form of the basis is not always available. We also remark that much of the discussion of the covariance in the discrete case carries through to the continuous case. For example, we can interpret $\mathbb{E}(\phi(x)\phi(y))$ as a covariance kernel and it is the Green's function of the domain with Dirichlet boundary conditions. Moreover, it has a logarithmic singularity as $x \to y$ in the bulk.

2. $1/f$ noise

$1/f$ noise or pink noise is a one-dimensional Gaussian stochastic process whose power spectral density is inversely proportional to its frequency. We will be informal in this section. Let $x : [0,\infty) \to \mathbb{R}$ be a random function such that for each t, $x(t)$ is a Gaussian random variable. Consider the truncated Fourier transform

$$\hat{x}_T(\omega) = \frac{1}{\sqrt{T}} \int_0^T x(t) e^{-i\omega t} dt. \tag{2.20}$$

The power spectral density is defined to be

$$S(\omega) = \lim_{T\to\infty} \mathbb{E}(|\hat{x}_T(\omega)|^2) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^T \mathbb{E}(x(t)x(t')) e^{i\omega(t-t')} dt dt'. \tag{2.21}$$

When the covariance of the process x is translation invariant, S is the Fourier transform of the autocorrelation function $\gamma(t) = \mathbb{E}(x(t_0)x(t+t_0))$ if the Fourier transform exists. $1/f$ noise is the situation where $S(\omega)$ is proportional to ω^{-1}. For more information, see [**33, 55**].

To see how this is related to logarithmically correlated Gaussian fields, consider the following construction of a Gaussian field on the unit circle ($[0, 2\pi)$ with periodic boundary conditions) [**33, 34, 32**]: let v_n be i.i.d. centered complex Gaussian random variables such that the imaginary part is an independent copy of the real part and $\mathbb{E}(|v_n|^2) = 1$ (note that also $\mathbb{E}(v_n^2) = 0$ with our assumptions) and define

$$V(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (v_n e^{int} + v_n^* e^{-int}). \tag{2.22}$$

Then

$$\begin{aligned}
\mathbb{E}(V(t)V(t')) &= \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(v_n^2 e^{in(t+t')} + |v_n|^2 e^{in(t-t')} + |v_n|^2 e^{in(t'-t)} + (v_n^*)^2 e^{-in(t+t')} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \left(e^{in(t-t')} + e^{in(t'-t)} \right) \\
&= \sum_{n\neq 0} \frac{1}{n} e^{in(t-t')} \\
&= -2\log\left| 2\sin\frac{t-t'}{2} \right|.
\end{aligned} \tag{2.23}$$

The third line says precisely that the power spectral density is proportional to the inverse of the frequency, i.e. that this is a model for $1/f$ noise. The last line says that we have logarithmic correlations.

3. White noise decompositions

In this section, we discuss Gaussian fields on $\mathbb{R}^d$ that can be formally written as

$$X(x) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} g(s, x, y) W(ds, dy), \tag{2.24}$$

where $g : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a suitable (deterministic) function and W is space-time white noise, i.e. a Gaussian process with short range correlations in space and time:

$$\mathbb{E}(W(ds,dy)W(ds',dy')) = \delta(s-s')\delta(y-y')dsdy. \tag{2.25}$$

We note that this is quite similar to our construction of the Gaussian Free Field and $1/f$ noise as linear combinations of i.i.d. Gaussians: on a discrete level $W(n,y)$ would just be i.i.d. standard Gaussians. If we took $g(s,x,y)$ proportional to $\delta_{y,0}\mathbf{1}\{s \geq 1\}$, we would have

$$X(x) = \sum_{n=1}^{\infty} g(n,x,0)W_{n,0}, \tag{2.26}$$

which is similar to the expansion we had for the GFF and $1/f$ noise.

There will be two main benefits for us from this type of a representation for the field. First of all, this will give a natural way to add an additional parameter to the field $X_t(x)$ such that as $t \to \infty$ $X_t(x) \to X(x)$:

$$X_t(x) = \int_{-\infty}^{t} \int_{\mathbb{R}^d} g(s,x,y)W(ds,dy). \tag{2.27}$$

Comparing this to the random series representation of the GFF, this would correspond to truncating the series, which we could interpret as the discretization of the GFF.

The second benefit will be that with a suitable choice of the function g, we will be able to build certain symmetry and scaling properties into the field.

To make rigorous this notion we introduce the notion of the isonormal Gaussian process [**45**]. Consider the (separable) Hilbert space $H = L^2(\mathbb{R} \times \mathbb{R}^d, ds \otimes dy)$ and the centered Gaussian process $(\eta(h))_{h \in H}$ with covariance $\mathbb{E}(\eta(h)\eta(g)) = (h,g)_{L^2(\mu)}$. Such a process can be constructed by choosing an orthonormal basis $(e_i)_{i=1}^{\infty}$ for H and taking $(a_i)_{i=1}^{\infty}$ i.i.d. standard Gaussians. Then for $h = \sum_{i=1}^{\infty} b_i e_i \in H$ one defines $\eta(h) = \sum_i a_i b_i$. Note that this is linear in h. This converges almost surely and in $L^2(\mathbb{P})$. Checking that the covariance is the correct one is simple. We then define

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} g(s,x,y)W(ds,dy) = \eta(g(\cdot,x,\cdot)). \tag{2.28}$$

Let us now consider a few specific choices of g which are common in current literature [**27, 28, 61, 8, 7, 5**] and we shall use when discussing Gaussian multiplicative chaos.

For $d=1$, consider $g_1(s,x,y) = \mathbf{1}\{|x-y| \leq \frac{1}{2}\min(e^{-s},1)\}e^{\frac{s}{2}}\mathbf{1}\{s \leq t\}$. $(s,y) \mapsto g_1(s,x,y)$ is in $L^2(\mathbb{R} \times \mathbb{R}^d, ds \otimes dy)$ for each x so $X_t(x) = \eta(g_1(\cdot,x,\cdot))$ is well defined. Let us calculate its covariance.

$$
\begin{aligned}
&\mathbb{E}(X_t(x)X_t(y)) \\
&= \int_{-\infty}^{t}\int_{\mathbb{R}} \mathbf{1}\left\{|x-z| \le \frac{1}{2}\min\left(e^{-s},1\right)\right\} \times \mathbf{1}\left\{|y-z| \le \frac{1}{2}\min\left(e^{-s},1\right)\right\} e^{s} ds dz \\
&= \int_{-\infty}^{0} e^{s}\int_{\mathbb{R}} \mathbf{1}\left\{|z-x| \le \frac{1}{2},\ |z-y| \le \frac{1}{2}\right\} dz ds \\
&+ \int_{0}^{t} e^{s}\int_{\mathbb{R}} \mathbf{1}\left\{|z-x| \le \frac{1}{2}e^{-s},\ |z-y| \le \frac{1}{2}e^{-s}\right\} dz ds \\
&= \mathbf{1}\{|x-y|<1\}(1-|x-y|) + \int_0^t e^s(e^{-s}-|x-y|)\mathbf{1}\{|x-y|<e^{-s}\}ds \\
&= \mathbf{1}\{|x-y|<1\}(1-|x-y|) + \int_0^{t\wedge \log\frac{1}{|x-y|}} (1-e^s|x-y|)ds \\
&= \begin{cases} 1+t-e^t|x-y|, & \text{for } |x-y| \le e^{-t} \\ -\log|x-y|, & \text{for } e^{-t} \le |x-y| \le 1 \\ 0, & \text{for} |x-y| \ge 1 \end{cases} .
\end{aligned}
$$

Consider now a slight modification of g_1: take $g_2 = \mathbf{1}\{s \ge 0\}g_1$. Again $X_t(x) = \eta(g_2(\cdot,x,\cdot))$ is well defined. This produces the covariance

$$
\mathbb{E}(X_t(x)X_t(y)) = \begin{cases} t-(e^t-1)|x-y|, & \text{for } |x-y| \le e^{-t} \\ -\log|x-y| + |x-y| - 1, & \text{for } e^{-t} \le |x-y| \le 1 \\ 0, & \text{for}|x-y| \ge 1 \end{cases} \tag{2.29}
$$

so the singularity is still logarithmic. We note that these two fields have a rather nice geometric interpretation: for the 'g_1-field' we sample a weighted white noise in the upper half plane over a vertical strip with a triangular tip at the real axis. For the 'g_2-field', we sample white noise over a triangle. This geometric interpretation can be used to visualize the covariance calculations and other constructions related to the fields.

Let us return to $\mathbb{R}^d$ and define $g_3(s,x,y) = g(e^s(x-y))e^{\frac{ds}{2}}\mathbf{1}\{s \in [0,t]\}$ for some $g \in L^2(\mathbb{R}^d)$. Since $g \in L^2$, $X_t(x) = \eta(g_3(\cdot,x,\cdot))$ is well defined. Let us write k for the convolution square of g: $k(x) = \int_{\mathbb{R}^d} g(x+y)g(y)dy$. We have in this situation

$$
\mathbb{E}(X_t(x)X_t(y)) = \int_0^t k(e^s(x-y))ds. \tag{2.30}
$$

We note that g_2 also fits this definition, but g_1 does not.

Let us consider what we need of k so that we have a logarithmic singularity in the covariance. After a change of variables, we have formally

$$
\mathbb{E}(X(x)X(y)) = \int_{|x-y|}^{\infty} \frac{k\left(u\frac{x-y}{|x-y|}\right)}{u} du. \tag{2.31}
$$

Assuming that k is nice enough (e.g. $k(x) \le |x|^{-\epsilon}$ as $x \to \infty$ or compact support), we see that any short range singularity in the covariance comes from the behavior of k near zero. To produce a logarithmic one, we simply need that k is non-zero (and finite which follows from $g \in L^2$) at zero, i.e. that $g \neq 0$.

Let us note that for all of the fields we have considered, there is a $\mathbf{1}\{s \leq t\}$ in the function g. This implies that for any t and $\epsilon > 0$, $X_{t+\epsilon} - X_t$ is independent of X_t (their covariance is zero since in the 'white noise expansion' one is proportional to $\mathbf{1}\{s \in [t, t+\epsilon)\}$ and the other to $\mathbf{1}\{s \leq t\}$ so their L^2 inner product is zero). Thus $t \mapsto X_t$ has independent increments. In fact, one can check that $t \mapsto X_t$ is continuous so since $\mathbb{E}((X_t(x) - X_0(x))^2) = ct$ for the fields we have considered, $X_t - X_0$ is Brownian motion.

Let us finish this section with a comment on the fields we have constructed. The white noise decomposition gives a nice heuristic picture of how we are thinking of these fields. The time parameter s always appears coupled to the spatial one in a form $e^s x$, so we can think of e^{-s} indexing the spatial scale we are at. Thus the white noise decomposition is writing the field as a weighted sum of independent Gaussians living on different spatial scales, where also the weight depends on the scale we are on. This again is similar to the expansions we had for the GFF and the $1/f$ noise.

4. Branching Random Walks and Branching Brownian motion

Let us assume that we are given a point process ξ on the real line and consider the following construction. At time $t = 0$, we have a single particle located at the origin. At time $t = 1$, this particle dies and gives birth to new particles whose locations are distributed according to the point process ξ. At time $t = 2$, each of these particles die and give birth to particles whose location with respect to the parent particle is distributed according to ξ (independently of the other parent particles). This process continues until no more offspring is produced (this may never happen). The collection of the locations of all the particles in all generations is called a branching random walk.

In this section we shall discuss some basic properties of a simple branching random walk. For us, the point process will always consist of two particles which are located at the same point and this point is distributed like a standard Gaussian. A similar and perhaps more common example (also used in [**10**]) is that where again there are two particles but the locations of the particles are independent of each other and each position is distributed like a standard Gaussian. At time n there are 2^n particles present. It will be convenient for us to index them by $\sigma \in \Sigma_n = \{0,1\}^n$. Write $(X_\sigma)_{\sigma \in \Sigma_n}$ for the positions of the particles present at time n in our first example and $(Y_\sigma)_{\sigma \in \Sigma_n}$ for the positions of the particles in our second example.

The relationship of the branching random walks to logarithmically correlated fields will come from giving the branching random walk a spatial structure through identifying σ with the binary expansion of a point, namely if we identify $\sigma \in \Sigma_n$ with a dyadic interval I_σ of length 2^{-n}, we set $X^n(x) = X_\sigma$ for $x \in I_\sigma$. It will be convenient for us to write this definition in a way that will allow a 'white noise expansion'. We write

$$X^n(x) = \sum_{\sigma \in \Sigma_n} \mathbf{1}\left\{x - \sum_{k=1}^{n} \sigma_k 2^{-k} \in [0, 2^{-n})\right\} X_\sigma \tag{2.32}$$

and

$$Y^n(x) = \sum_{\sigma\in\Sigma_n} \mathbf{1}\left\{x - \sum_{k=1}^n \sigma_k 2^{-k} \in [0, 2^{-n})\right\} Y_\sigma, \tag{2.33}$$

where we used the notation $\sigma = (\sigma_1, ..., \sigma_n)$. Note that only one σ contributes to the sum for each x so this is indeed identifying σ with a dyadic interval of length n and defining $X^n(x)$ to be X_σ on this interval. This type of fields are known as hierarchical models in physics and in the physics literature go back to Dyson [**30**]. They are used as simplifications of more complicated models especially in the industry of renormalization. In some cases, they are known to capture some of the relevant properties of the model they are approximating (see e.g. references in [**39**]).

Let us write for $\sigma = (\sigma_1, ..., \sigma_n)$, $\sigma|k = (\sigma_1, ..., \sigma_k)$. We can then write X_σ as $\sum_{k=1}^n V_{\sigma|k}$, where $V_{\sigma|k}$ are standard Gaussians (for Y_σ they would be independent, but for X_σ, neighboring ones in the dyadic sense are identical so $\mathbb{E}(V_{\sigma|k}V_{\sigma'|k'}) = \delta_{k,k'}\delta_{\sigma|k-1,\sigma'|k-1}$). Plugging this into the definition of $X^n(x)$, we have

$$X^n(x) = \sum_{k=1}^n \sum_{\sigma\in\Sigma_n} \mathbf{1}\left\{x - \sum_{j=1}^n \sigma_j 2^{-j} \in [0, 2^{-n})\right\} V_{\sigma|k}. \tag{2.34}$$

If we interpret σ as a spatial variable (through the binary expansion), we see that this is simply a white noise expansion where k encodes the spatial scale on which the Gaussians live (actually we are not summing over independent random variables, but if we we write the sum over σ as a sum over $\sigma|k \in \Sigma_k$ and the sum over descendants of $\sigma|k$ and perform the sum over descendants and sum over neighboring particles at level k in the dyadic sense, we get a sum over independent standard Gaussians which we can view as a white noise decomposition).

Let us now calculate the covariance of the field to see if it bears any further resemblance to the fields we considered in the previous sections.

$$\begin{aligned}
\mathbb{E}(X^n(x)X^n(y)) &= \sum_{k_1,k_2=1}^n \sum_{\sigma,\sigma'\in\Sigma_n} \mathbf{1}\left\{x - \sum_{j=1}^n \sigma_j 2^{-j} \in [0, 2^{-n})\right\} \\
&\quad \times \mathbf{1}\left\{y - \sum_{j=1}^n \sigma_j' 2^{-j} \in [0, 2^{-n})\right\} \mathbb{E}(V_{\sigma|k_1}V_{\sigma'|k_2}) \\
&= \sum_{k=1}^n \sum_{\sigma,\sigma'\in\Sigma_n} \mathbf{1}\left\{x - \sum_{j=1}^n \sigma_k 2^{-j} \in [0, 2^{-n})\right\} \\
&\quad \times \mathbf{1}\left\{y - \sum_{j=1}^n \sigma_k' 2^{-j} \in [0, 2^{-n})\right\} \mathbf{1}\{\sigma|k-1 = \sigma'|k-1\}.
\end{aligned} \tag{2.35}$$

So we see that for a fixed k, for the summand to be non-zero, we need that the first $k-1$ digits in the binary expansion of the numbers x and y must be equal. This means that we only sum over k for which $|x-y| \le 2^{-k+1}$. Moreover, there is only one term in the σ, σ' sum (since there is only one dyadic interval of length 2^{-n} containing x and one containing y). So we can estimate the sum upwards to

$$\mathbb{E}(X^n(x)X^n(y)) \leq \sum_{k=1}^{n\wedge\left(\frac{-\log|x-y|}{\log 2}+1\right)} 1 = \left(-\frac{\log|x-y|}{\log 2}+1\right)\wedge n. \tag{2.36}$$

Moreover, the variance of $X^n(x)$ is n. A similar estimate holds for $Y^n(x)$ as well. So we have constructed a field $X^n(x)$ that can be represented through a white noise decomposition similar in spirit to the ones we considered in the previous section and it is comparable to the fields $X_n(x)$ of the previous section at least in the sense that their variances are of the same order and the correlations of X^n are bounded from above by the correlations of X_n (up to a bounded term). Due to this relationship of the covariances, one can expect to use results similar to Slepian's lemma to estimate functionals of one field in terms of functionals of the other.

We note that X^n differs from X_n in a significant way: there are points that are near to each other in the Euclidean distance, but have very little correlation for X^n - consider points close to $\frac{1}{2}$ but on opposite sides of it. We remark that the covariance of the field can be related to $\log d(x,y)$ for the ultrametric distance on the unit interval, but we shall not make use of this any further. We also note that fields can be constructed on higher dimensional unit hypercubes as well if one considers more complicated branching: for example a branching random walk branching into four particles at each time can be similarly interpreted as a field on the unit square and so on.

We finish this section with a short discussion about a continuum limit of this process. As the scaling limit of a random walk is Brownian motion, one might want to consider a branching Brownian motion. The precise definition is the following. Consider an exponentially distributed random variable with rate one. Run a Brownian motion started at the origin until this exponential time. At this time, kill the particle and start two independent Brownian motions from the original particle's position. These new particles behave as the original one and independently from each other. A more complicated branching structure with a random number of offspring and a different branching rate are also possible.

5. Tools for analysis

For calculating expectations of simple functionals (such as first moments) that don't notice the correlations, one would want to make use of the fact that $X_t(x)$ is a Brownian motion and $X^n(x)$ is a random walk. This indeed is possible. These type of theorems are known as many-to-one theorems. The proof is a simple change of variables in the branching random walk case and an application of Girsanov's theorem in the continuous 'time' fields. We shall only state and prove the result for the branching random walk X_σ, but generalizations to other cases are straight forward.

THEOREM 2.1. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Then*

$$\begin{aligned}&\mathbb{E}\left(\sum_{\sigma\in\Sigma_n} g(X_{\sigma|1}-\sqrt{2\log 2}, X_{\sigma|2}-2\sqrt{2\log 2}, ..., X_\sigma - n\sqrt{2\log 2})\right)\\ &\quad = \mathbb{E}(e^{-\sqrt{2\log 2}S_n} g(S_1, ..., S_n)),\end{aligned}$$

where $\{S_k\}$ is a random walk with standard Gaussian increments.

PROOF. This is a direct calculation:

$$\begin{aligned}
&\mathbb{E}\left(\sum_{\sigma\in\Sigma_n} g(X_{\sigma|1}-a, X_{\sigma|2}-2a, ..., X_\sigma - na)\right)\\
&= 2^n \int_{\mathbb{R}^n} \prod_{k=1}^n \left(dy_k \frac{e^{-\frac{y_k^2}{2}}}{\sqrt{2\pi}}\right) g(y_1-\sqrt{2\log 2}, ..., y_1+...+y_n - n\sqrt{2\log 2})\\
&= e^{n\log 2}\int_{\mathbb{R}^n} \prod_{k=1}^n \left(dy_k \frac{e^{-\frac{(y_k+\sqrt{2\log 2})^2}{2}}}{\sqrt{2\pi}}\right) g(y_1, ..., y_1+...+y_n)\\
&= e^{n\log 2}\int_{\mathbb{R}^n} \prod_{k=1}^n \left(dy_k \frac{e^{-\frac{y_k^2}{2}}}{\sqrt{2\pi}}\right) e^{-\sqrt{2\log 2}\sum_{k=1}^n y_k} e^{-n\log 2} g(y_1, ..., y_1+...+y_n)\\
&= \mathbb{E}(e^{-\sqrt{2\log 2}S_n} g(S_1, ..., S_n)).
\end{aligned}$$

□

Calculating more complicated objects, such as second moments, which see the correlations in the field, may be very difficult. This is one of the main benefits of the branching random walk - its hierarchical structure allows a simpler treatment of second moments. We state the following theorem whose proof is again essentially just a change of variables. We shall again only state and prove it for X_σ, but a similar result holds for Y_σ, branching Brownian motions and more complicated branching random walks.

THEOREM 2.2. *Let $f:\mathbb{R}^n\to\mathbb{R}$ and $g:\mathbb{R}^n\to\mathbb{R}$ be measurable functions. Then*

$$\begin{aligned}
&\sum_{\sigma\in\Sigma_n, \sigma'\in\Sigma_n} \mathbb{E}\Big(f(X_{\sigma|1}-\sqrt{2\log 2}, ..., X_\sigma - \sqrt{2\log 2}n)\\
&\qquad \times g(X_{\sigma'|1}-\sqrt{2\log 2}, ..., X_{\sigma'}-\sqrt{2\log 2}n)\Big)\\
&= \sum_{l=0}^{n-1} \mathbb{E}(e^{-\sqrt{2\log 2}S_n} e^{-\sqrt{2\log 2}S'_{n-l-1}} f(S_1, ..., S_n)\\
&\qquad \times g(S_1, ..., S_{l+1}, S_{l+1}+S'_1, ..., S_l + S'_{n-l-1})),
\end{aligned}$$

where S and S' are independent random walks with standard Gaussian increments.

PROOF. Let l be the level of the last common ancestor of σ and σ' (write $a(\sigma,\sigma')=l$ for this) so that $X_{\sigma|l+1}=X_{\sigma'|l+1}$. Thus for $k>l+1$, $X_{\sigma|k}-X_{\sigma|l+1}$ and $X_{\sigma'|k}-X_{\sigma'|l+1}$ are independent and we have

$$
\begin{aligned}
&\sum_{\sigma\in\Sigma_n,\sigma'\in\Sigma_n} \mathbb{E}\Big(f(X_{\sigma|1}-\sqrt{2\log 2},...,X_\sigma-\sqrt{2\log 2}n) \\
&\qquad \times g(X_{\sigma'|1}-\sqrt{2\log 2},...,X_{\sigma'}-\sqrt{2\log 2}n)\Big) \\
&=\sum_{l=0}^{n-1}\sum_{\sigma\in\Sigma_n,\sigma'\in\Sigma_n} \mathbf{1}\{a(\sigma,\sigma')=l\}\mathbb{E}\Big(f(X_{\sigma|1}-\sqrt{2\log 2},...,X_\sigma-\sqrt{2\log 2}n) \\
&\times g(X_{\sigma|1}-\sqrt{2\log 2},...,X_{\sigma|l+1}-\sqrt{2\log 2}(l+1), \\
&X_{\sigma|l+1}-\sqrt{2\log 2}(l+1)+(X_{\sigma'|l+2}-X_{\sigma|l+1}-\sqrt{2\log 2}),..., \\
&X_{\sigma|l+1}-\sqrt{2\log 2}(l+1)+(X_{\sigma'}-X_{\sigma|l+1}-\sqrt{2\log 2}(n-l-1)))\Big).
\end{aligned}
$$

Writing this out as an integral performing the same shifts in the integration variables and noting that $\sum_{\sigma\in\Sigma_n,\sigma'\in\Sigma_n}\mathbf{1}\{a(\sigma,\sigma')=l\}=2^n2^{n-l-1}$, we find the statement through similar reasoning as in the previous result. □

Once we have transformed branching random walk quantities into random walk ones, we shall need some estimates concerning a random walk. In particular, we shall need estimates of the probability that a Gaussian random walk starting at x is at the point y at time n and stays positive during this time. For this, we define the discrete time Brownian bridge over $\{0,...,n\}$ and call the process $\{Y_k\}_{k=0}^n$. This is simply the continuous time Brownian bridge sampled at integer times. We will write $\mathbb{P}_n^{x,y}$ for the law of such a process conditioned to start at x and terminate at y. The main estimate we shall need for it, is the following gambler's ruin estimate:

Proposition 2.3. *Let $\{Y_k\}_{k=0}^n$ be a discrete time Brownian bridge. Then for $0\le x,y\le\sqrt{n}$ and some positive constants C_1 and C_2*

$$
C_1\frac{xy}{n}\le\mathbb{P}_n^{x,y}(Y_k>0\text{ for all }k\le n)\le C_2\frac{(1+x)(1+y)}{n}. \tag{2.37}
$$

Here $\mathbb{P}_n^{x,y}$ is the law of a discrete time Brownian bridge from x to y over $\{0,...,n\}$. The upper bound holds for any positive values of x and y.

The proof is quite simple given the gambler's ruin estimate for a random walk (see e.g. [**50**]): let S be a centered random walk satisfying some (very non-restrictive - in particular the Gaussian case satisfies them) regularity conditions, then there is a constant $C>0$ such that for $0\le x\le\sqrt{n}$

$$
\mathbb{P}^x(S_k\ge 0\text{ for }k\le n)\le C\frac{x+1}{\sqrt{n}}. \tag{2.38}
$$

This holds for larger x as well of course. The basic idea of the proof for the bridge is to split the bridge into three parts of equal length, then estimate upwards the probability by forgetting what happens in the middle part so we simply get the product of two random walk estimates. The lower bound comes from estimating downwards to the corresponding probability for a continuous time Brownian bridge. The distribution of the maximum of the Brownian bridge is a known result (see e.g. [**18, 46**]) and the lower bound comes from this. For the details of the discrete bridge estimate, we refer to [**68**].

We will also need a lower bound of the form $\frac{(1+x)(1+y)}{n}$, i.e. for $x = y = 0$, we get a lower bound of order n^{-1}. This is simple to obtain using the bound we have by noting that in fact

$$\mathbb{P}_n^{0,0}(Y_k \geq 0, \text{ for } k \leq n) = \mathbb{P}_n^{0,0}(Y_k \geq 0, \text{ for } k \in \{1, ..., n-1\}), \tag{2.39}$$

conditioning on Y_1 and Y_{n-1} and using the estimate for the bridge from Y_1 to Y_{n-1}.

5.1. Some results from renewal theory. In addition to the previous results, we shall need some standard results from renewal theory and we shall give a brief review of them here. For a more in-depth review, see [**31, 45**].

In its classical form, renewal theory is the study of random walks with almost surely non-negative increments with finite positive expectation. We will use this to study random walks with centered increments. Indeed, consider a general one-dimensional random walk S (we shall assume in this chapter that $S_0 = 0$) and introduce the ascending ladder times τ_k through the recursion $\tau_0 = 0$ and

$$\tau_n = \inf\{k > \tau_{n-1} : \ S_k > S_{\tau_{n-1}}\}. \tag{2.40}$$

We then define $x_n = S_{\tau_n} - S_{\tau_{n-1}}$. We note that clearly τ_n is a stopping time so by the strong Markov property of the random walk, for each n $(S_{\tau_n+k} - S_{\tau_n})_k$ is distributed like (S_k). Thus for each n, $\tau_n - \tau_{n-1}$ is distributed like τ_1 and x_n is distributed like x_1. So we see that

$$X_n = \sum_{k=0}^{n} x_n = S_{\tau_n} \tag{2.41}$$

is a random walk with non-negative increments. If we assume that S is centered and its increments have finite positive variance, one can show that $\mathbb{E}(S_{\tau_1})$ is finite and positive.

One of the central objects of the theory is the renewal function (which is finite by our assumptions)

$$R(u) = \sum_{n=0}^{\infty} \mathbb{P}(X_n \leq u) = \sum_{n=0}^{\infty} \mathbb{P}(S_{\tau_n} \leq u). \tag{2.42}$$

An elementary, but useful result is the renewal theorem (we refer to the literature for a proof):

Theorem 2.4. *There exists a constant $c \in (0, \infty)$ such that*

$$\lim_{u \to \infty} \frac{R(u)}{u} = c. \tag{2.43}$$

We also make use of a dual representation of the renewal function: let $\tau = \inf\{k \geq 1 : S_k \leq 0\}$ and define the function

$$\tilde{R}(u) = \mathbb{E}\left(\sum_{j=0}^{\tau-1} \mathbf{1}\{S_j \leq u\}\right). \tag{2.44}$$

Making use of the remark that for a random walk, $(S_1, S_2, ..., S_n) \stackrel{d}{=} (S_n - S_{n-1}, S_n - S_{n-2}, ..., S_n)$, it is easy to check that $\tilde{R} = R$.

Using this representation, we can prove that R satisfies a certain integral equation:

THEOREM 2.5. *Under some very loose regularity conditions for S (namely we require that $\mathbb{P}(\tau < \infty) = 1$ which holds for example if the Gambler's ruin estimate holds)*

$$R(u) = \mathbb{E}(R(u - S_1)\mathbf{1}\{S_1 \le u\}). \tag{2.45}$$

PROOF. Since we assume that $\mathbb{P}(\tau < \infty) = 1$,

$$\begin{aligned}
\mathbb{E}(R(u - S_1)\mathbf{1}\{S_1 \le u\}) &= \mathbb{E}\left(\sum_{j=0}^{\infty} \mathbf{1}\{S_1 + \tilde{S}_j \le u,\ \tilde{S}_1 > 0, ..., \tilde{S}_j > 0\}\right) \\
&= \mathbb{E}\left(\sum_{j=0}^{\infty} \mathbf{1}\left\{\tilde{S}_j + S_1 \le u,\ \tilde{S}_1 > 0, ..., \tilde{S}_j > 0, \tilde{S}_j + S_1 > 0\right\}\right) \\
&+ \mathbb{E}\left(\sum_{j=0}^{\infty} \mathbf{1}\left\{\tilde{S}_j + S_1 \le 0,\ \tilde{S}_1 > 0, ..., \tilde{S}_j > 0\right\}\right) \\
&= \mathbb{E}\left(\sum_{j=1}^{\infty} \mathbf{1}\left\{S_j \le u,\ S_1 > 0, ..., S_j > 0\right\}\right) + \mathbb{E}\left(\sum_{j=0}^{\infty} \mathbf{1}\left\{\tau = j+1\right\}\right) \\
&= \mathbb{E}\left(\sum_{j=1}^{\infty} \mathbf{1}\left\{S_j \le u,\ S_1 > 0, ..., S_j > 0\right\}\right) + \mathbb{P}(\tau < \infty) \\
&= \mathbb{E}\left(\sum_{j=1}^{\infty} \mathbf{1}\left\{S_j \le u,\ S_1 > 0, ..., S_j > 0\right\}\right) + 1 \\
&= R(u)
\end{aligned}$$

□

CHAPTER 3

Gaussian multiplicative cascades and Gaussian multiplicative chaos

In this section, we shall discuss some of the history and current theory of rigorously defining measures of the form $e^{\beta X(x)}\sigma(dx)$, where X is a Gaussian field and σ is a Radon measure. We will first focus on the situation when X is a branching random walk field and σ the Lebesgue measure on $[0,1]$ and then consider the more general situation.

1. Multiplicative cascades

When X is a branching random walk field, these type of measures are known as Mandelbrot (or multiplicative) cascades and were introduced by Mandelbrot [**53**] as a toy model exhibiting similar fractal and statistical properties as those appearing in turbulence. The rigorous study of them was begun by Kahane and Peyrière [**42, 44, 58**].

We'll state their main convergence result in terms of the field X^n. Similar results hold for Y^n and more general branching random walk fields.

THEOREM 3.1. *For $\beta < \sqrt{2\log 2}$, the measures*

$$\mu_{\beta,n}(dx) = e^{\beta X^n(x) - \frac{\beta^2}{2}\mathbb{E}((X^n(x))^2)}dx \tag{3.1}$$

converge weakly almost surely to a non-trivial non-atomic measure which is singular with respect to the Lebesgue measure. For $\beta \geq \sqrt{2\log 2}$, the measures converge to zero almost surely.

Let us discuss some of the simpler parts of the proof in the case of X^n. We remark that the exponential term $e^{-\frac{\beta^2}{2}\mathbb{E}((X^n(x))^2)}$ is simply there to normalize the expectation of the mass of the measure to one. We also note that the field $X^n(x)$ enjoys a nice scaling property that the measure inherits. Recall the definition of $X^n(x)$ and let $0 < m < n$. Let us also write x_m for the first m terms in the dyadic expansion of x and define $\tilde{x} = 2^m(x - x_m) \in [0,1)$.

$$\begin{aligned}
X^n(x) &= \sum_{\sigma\in\Sigma_n} \mathbf{1}\left\{x - \sum_{k=1}^{n}\sigma_k 2^{-k} \in [0,2^{-n})\right\} X_\sigma \\
&= \sum_{\sigma'\in\Sigma_m} \sum_{\substack{\sigma\in\Sigma_n:\\ \sigma|m=\sigma'}} \mathbf{1}\left\{x - \sum_{k=1}^{m}\sigma'_k 2^{-k} - \sum_{k=m+1}^{n}\sigma_k 2^{-k} \in [0,2^{-n})\right\}(X_{\sigma'} + X_\sigma - X_{\sigma|m}).
\end{aligned}$$

We now remark that we can write

$$\mathbf{1}\left\{x-\sum_{k=1}^{m}\sigma'_k 2^{-k}+\sum_{k=m+1}^{n}\sigma_k 2^{-k}\in[0,2^{-n})\right\}$$
$$=\mathbf{1}\left\{x-\sum_{k=1}^{m}\sigma'_k 2^{-k}\in[0,2^{-m})\right\}\mathbf{1}\left\{\tilde{x}-\sum_{k=m+1}^{n}\sigma_k 2^{-(k-m)}\in[0,2^{-(n-m)})\right\}.$$

Thus noting that $(X_\sigma - X_{\sigma|m})_{\sigma\in\Sigma_n:\sigma|m=\sigma'}$ is distributed like $(X_\sigma)_{\sigma\in\Sigma_{n-m}}$ and independent of $(X_{\sigma'})_{\sigma'\in\Sigma_m}$

$$X^n(x)\overset{d}{=}X^m(x)+\tilde{X}^{n-m}(\tilde{x}), \tag{3.2}$$

where $\tilde{X}^{n-m}$ is independent of X^m and distributed like X^{n-m}. Consider then a dyadic interval I_σ of length 2^{-m}. We see by this decomposition that

$$\mu_{\beta,n}(I_\sigma)\overset{d}{=}\int_{I_\sigma}e^{\beta X^m(x)-\frac{\beta^2}{2}\mathbb{E}((X^m(x))^2}e^{\beta\tilde{X}^{n-m}(\tilde{x})-\frac{\beta^2}{2}\mathbb{E}((\tilde{X}^{n-m}(\tilde{x}))^2}dx. \tag{3.3}$$

We note that $X^m(x)$ is simply the constant X_σ for $x\in I_\sigma$. Thus making a change of variable in the integration: $y=\tilde{x}$, we find

$$\mu_{\beta,n}(I_\sigma)\overset{d}{=}2^m e^{\beta X_\sigma-\frac{\beta^2}{2}\mathbb{E}(X_\sigma^2)}\int_0^1 e^{\beta\tilde{X}^{n-m}(y)-\frac{\beta^2}{2}\mathbb{E}((\tilde{X}^{n-m}(y))^2)}dy \tag{3.4}$$

$$=2^m e^{\beta X_\sigma-\frac{\beta^2}{2}\mathbb{E}(X_\sigma^2)}\tilde{\mu}_{\beta,n-m}([0,1)), \tag{3.5}$$

where $\tilde{\mu}_{\beta,n-m}$ is distributed like $\mu_{\beta,n-m}$ and independent of $(X_\sigma)_{\sigma\in\Sigma_m}$. Going through our arguments again, we note that we can do this simultaneously for all dyadic intervals of length 2^{-m}:

$$(\mu_{\beta,n}(I_\sigma))_{\sigma\in\Sigma_m}\overset{d}{=}\left(2^m e^{\beta X_\sigma-\frac{\beta^2}{2}\mathbb{E}(X_\sigma^2)}\mu^\sigma_{\beta,n-m}([0,1))\right)_{\sigma\in\Sigma_m}, \tag{3.6}$$

where $\mu^\sigma_{\beta,n-m}([0,1))$ are independent copies of $\mu_{\beta,n-m}([0,1))$ and independent of $(X_\sigma)_{\sigma\in\Sigma_m}$. So we see that for any set A which is a finite union of dyadic intervals, we can write $\mu_{n,\beta}(A)$ (for large enough n) as a sum consisting of exponentials of a branching random walk whose length depends only on the set A and not of n and independent copies of $\mu_{n-k}([0,1))$ for some k depending on the set A and the number of these copies also depends only on the set A and not on n. We conclude that to show that the measures $\mu_{n,\beta}$ converge, we only need to show that

$$W_{n,\beta}=\mu_{n,\beta}([0,1))=2^{-n}\sum_{\sigma\in\Sigma_n}e^{\beta X_\sigma-\frac{\beta^2}{2}\mathbb{E}(X_\sigma^2)}=\sum_{\sigma\in\Sigma_n}e^{\beta\left(X_\sigma-\left(\frac{\beta}{2}+\frac{\log 2}{\beta}\right)n\right)} \tag{3.7}$$

converges.

The convergence of this object is something that has also been noted early on in the study of general branching random walks [**15**]. In this field it is known as the additive martingale. It is indeed a martingale and this is the reason it converges. To see that it is a martingale, we write any σ' in Σ_{n+1} as (σ,i) where σ runs through Σ_n and $i\in\{0,1\}$ and note that $X_{\sigma'}=X_\sigma+y_{(\sigma,i)}$. In fact $y_{(\sigma,0)}=y_{(\sigma,1)}=: y_\sigma$

and is a standard Gaussian. Moreover, y_σ is independent of $(X_\sigma)_{\sigma\in\Sigma_n}$ and $y_{\sigma'}$ for $\sigma\neq\sigma'$. Thus we find

(3.8)
$$W_{n+1,\beta}=\sum_{\sigma\in\Sigma_n}\sum_{i=0}^{1}e^{\beta\left(X_\sigma+y_\sigma\right)-\left(\frac{\beta}{2}+\frac{\log 2}{\beta}\right)(n+1)\big)}=\sum_{\sigma\in\Sigma_n}e^{\beta y_\sigma-\frac{\beta^2}{2}}e^{\beta\left(X_\sigma-\left(\frac{\beta}{2}+\frac{\log 2}{\beta}\right)n\right)}.$$

By independence of y_σ of (X_σ), we see that conditioning on $(X_\sigma)_{\sigma\in\Sigma_k}$ for $k\leq n$ (call this σ-algebra $\mathcal{F}_n$), $\mathbb{E}(W_{n+1,\beta}|\mathcal{F}_n)=W_{n,\beta}$, i.e. $W_{n,\beta}$ is a martingale. Moreover, $W_{n,\beta}\geq 0$ so by the martingale convergence theorem, $W_{n,\beta}$ converges almost surely to a non-negative random variable.

We will not go further into questions like uniform integrability required for the non-triviality of the limit since we will give an outline of an alternative proof for convergence to a non-trivial object in the next section. Also we shall not discuss non-atomicity since this also follows from the modulus of continuity estimates in [**10**].

This concludes the classical part of the theory of multiplicative cascades which has been around for over 30 years. After this, there has been much work (in many cases independent) in various fields on the question of what actually happens for $\beta\geq\sqrt{2\log 2}$. It turns out that there exists a deterministic normalization $f_{n,\beta}$ so that $f_{n,\beta}\mu_{n,\beta}$ converges to something non-trivial. This question was also suggested by Mandelbrot [**53**]. Indeed, in [**3**], it was proved (for very general branching random walks but we state the result in the notation of X^n) $\sqrt{n}W_{n,\sqrt{2\log 2}}$ converges in probability to a non-trivial random variable. In [**52**], it was shown (again for very general branching random walk, but we state the result corresponding to X^n) that $n^{\frac{3\beta}{2\sqrt{2\log 2}}}e^{\frac{1}{2}(\beta-\sqrt{2\log 2})^2 n}W_{n,\beta}$ converges in distribution to something non-trivial. In fact already in [**18**] it was proved that the corresponding objects for a branching Brownian motion converge in distribution. In [**23**] it was noted that this convergence result should hold for the branching random walk as well. Based on these two papers (following [**18**] closely) it was proven in [**68**] that up to a bounded deterministic normalization, these objects converge in distribution for the field X^n. Similar results up to bounded factors were also found already in [**37**] In addition, as noted in the first chapter, the $\beta\to\infty$ case corresponds to studying the distribution of the maximum of the variables X_σ. This has also been studied extensively [**1, 2, 4**]. Finally it was noted in [**12**] that the convergence of the total mass $f_{n,\beta}\mu_{n,\beta}([0,1))$ implies the convergence of the entire measure (the mode of convergence being the same as for the total mass). They also expressed the limiting measures for $\beta>\sqrt{2\log 2}$ in terms of the limiting measure for $\beta=\sqrt{2\log 2}$ and a stable Lévy subordinator implying that the limiting measure is purely atomic for $\beta>\sqrt{2\log 2}$. Finally in [**10**] it was proven that for $\beta=\sqrt{2\log 2}$, the limiting measure is non-atomic. Let us sum these results up as a theorem.

Theorem 3.2. *The measure*

$$\sqrt{n}\mu_{\sqrt{2\log 2},n}(dx)=\sqrt{n}e^{\sqrt{2\log 2}X^n(x)-n\log 2}dx \tag{3.9}$$

converges weakly in probability to a non-trivial almost surely non-atomic random Borel measure $\mu_{\sqrt{2\log 2}}$. For $\beta>\sqrt{2\log 2}$, there exists a positive constant c_β such that

$$n^{\frac{3\beta}{2\sqrt{2\log 2}}} e^{\frac{1}{2}(\beta-\sqrt{2\log 2})^2 n} \mu_{\beta,n}(dx) = n^{\frac{3\beta}{2\sqrt{2\log 2}}} e^{\frac{1}{2}(\beta-\sqrt{2\log 2})^2 n} e^{\beta X^n(x) - \frac{\beta^2}{2} n} dx \tag{3.10}$$

converges weakly in distribution to a measure μ_β which can be written as

$$\mu_\beta([0,t]) \stackrel{d}{=} c_\beta L_{\frac{\sqrt{2\log 2}}{\beta}}(\mu_{\sqrt{2\log 2}}([0,t])), \tag{3.11}$$

where L_α is a stable Lévy subordinator of index α independent of $\mu_{\sqrt{2\log 2}}$, i.e. a non-negative pure jump process whose Laplace transform is given by $\mathbb{E}(e^{-tL^\alpha(s)}) = e^{-st^\alpha}$. L_α being a pure jump process, μ_β is purely atomic.

We now point out the similarity with the REM. As in the case of independent random variables, the low temperature measure is given by composing a stable Lévy subordinator with the critical measure, but the difference is now that the critical measure is not the Lebesgue measure, but a random measure.

In fact at criticality, there is a another way to construct the limiting measure through a random normalization of the critical measure. We shall also state this as a theorem and then discuss some simple aspects of the proof and the history.

Theorem 3.3. *There is a deterministic constant c such that the signed measure*

$$\nu_n(dx) = (\sqrt{2\log 2}n - X^n(x)) e^{\sqrt{2\log 2} X^n(x) - n\log 2} dx \tag{3.12}$$

converges weakly almost surely to $c\mu_{\sqrt{2\log 2}}(dx)$.

We note that in this case, the scaling relation of the field X^n implies through similar arguments as before that for a dyadic interval I_σ of length 2^{-m}

$$\begin{aligned} \nu_n(I_\sigma) &\stackrel{d}{=} 2^m(\sqrt{2\log 2}m - X_\sigma) e^{\sqrt{2\log 2}X_\sigma - m\log 2} \tilde{\mu}_{\sqrt{2\log 2}, n-m}([0,1)) \\ &+ 2^m e^{\sqrt{2\log 2}X_\sigma - m\log 2} \tilde{\nu}_{n-m}([0,1)), \end{aligned} \tag{3.13}$$

where $\tilde{\nu}_{n-m}$ is an independent copy of ν_{n-m} which is independent of everything besides $\tilde{\mu}_{\sqrt{2\log 2}, n-m}$. We note that as $n \to \infty$, the first terms goes to zero, by the classical result so we only have to worry about the second one. Again, we can do this for all dyadic intervals of length 2^{-m} at the same time so to prove the convergence of the measure $\nu_n(dx)$, it is enough to prove that

$$D_n = \nu_n([0,1)) = \sum_{\sigma \in \Sigma_n} (\sqrt{2\log 2}n - X_\sigma) e^{\sqrt{2\log 2}X_\sigma - 2n\log 2} \tag{3.14}$$

converges.

This is a well known question in the theory of branching random walk and branching Brownian motion (see [**16, 48**] and references therein). The object is known as the derivative martingale and its convergence again comes from martingale theory. While it is elementary to check that it is a martingale, it is not strictly positive so its convergence is not immediately clear.

To construct a positive martingale similar to D_n, let $\alpha \geq 0$ and R be the renewal function for the random walk which each branch performs (so in the case of X^n the standard Gaussian random walk). Consider

(3.15)
$$D_n^{(\alpha)}=\sum_{\sigma\in\Sigma_n}R(\alpha+\sqrt{2\log 2}n-X_\sigma)e^{\sqrt{2\log 2}X_\sigma-2n\log 2}\mathbf{1}\{\alpha\geq X_{\sigma|k}-\sqrt{2\log 2}k,\ k\leq n\}.$$

To see that this is a martingale, use a similar decomposition as in proving that $W_{n,\beta}$ is a martingale to see that

$$\begin{aligned}D_{n+1}^{(\alpha)}=\sum_{\sigma\in\Sigma_n}&2e^{\sqrt{2\log 2}y_\sigma-2\log 2}R(\alpha+\sqrt{2\log 2}n-X_\sigma+\sqrt{2\log 2}-y_\sigma)\\&\times e^{\sqrt{2\log 2}X_\sigma-2n\log 2}\mathbf{1}\{\alpha\geq X_{\sigma|k}-\sqrt{2\log 2}k,\ k\leq n\}\\&\times\mathbf{1}\{\alpha\geq X_\sigma-\sqrt{2\log 2}n+y_\sigma-\sqrt{2\log 2}\}\end{aligned}$$

and after a shift of the integration variable y_σ

$$\begin{aligned}\mathbb{E}(D_{n+1}^{(\alpha)}|\mathcal{F}_n)&=\sum_{\sigma\in\Sigma_n}e^{\sqrt{2\log 2}X_\sigma-2n\log 2}\mathbf{1}\{\alpha\geq X_{\sigma|k}-\sqrt{2\log 2}k,\ k\leq n\}\\&\times\int_{\mathbb{R}}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}R(\alpha+\sqrt{2\log 2}n-X_\sigma-y)\mathbf{1}\{\alpha+\sqrt{2\log 2}n-X_\sigma\geq y\}\\&=\sum_{\sigma\in\Sigma_n}e^{\sqrt{2\log 2}X_\sigma-2n\log 2}\mathbf{1}\{\alpha\geq X_{\sigma|k}-\sqrt{2\log 2}k,\ k\leq n\}\\&R(\alpha+\sqrt{2\log 2}n-X_\sigma)\\&=D_n^{(\alpha)},\end{aligned}$$

where we used Theorem 2.5. So $D_n^{(\alpha)}$ is a positive martingale and it converges.

Let us now note that since $e^{\sqrt{2\log 2}(\max_{\sigma\in\Sigma_n}X_\sigma-\sqrt{2\log 2}n)}\leq W_{\sqrt{2\log 2},n}\to 0$ almost surely, $\max_{\sigma\in\Sigma_n}X_\sigma-\sqrt{2\log 2}n\to-\infty$ almost surely. Consider then the event A_α such that $X_\sigma-\sqrt{2\log 2}|\sigma|\leq\alpha$ for all $\sigma\in\cup_k\Sigma_k$ (here we write $|\sigma|=k$ for $\sigma\in\Sigma_k$). By our reasoning here $\mathbb{P}(A_\alpha)\to 1$ as $\alpha\to\infty$. Moreover, let $C=\lim_{u\to\infty}\frac{R(u)}{u}$ (which exists and is positive by Theorem 2.4). Now on A_α,

$$\begin{aligned}\lim_{n\to\infty}(D_n^{(\alpha)}-CD_n)&=\lim_{n\to\infty}\sum_{\sigma\in\Sigma_n}(R(\alpha+\sqrt{2\log 2}n-X_\sigma)-C(\sqrt{2\log 2}n-X_\sigma))\\&\times e^{\sqrt{2\log 2}X_\sigma-2n\log 2}\\&=a\lim_{n\to\infty}W_{\sqrt{2\log 2},n}\\&=0.\end{aligned}$$

Here we used the fact that $\max_{\sigma\in\Sigma_n}X_\sigma-\sqrt{2\log 2}n\to-\infty$ almost surely and that $\frac{R(u)}{u}\to C$ as $u\to\infty$. So we conclude that D_n also converges almost surely on A_α. D_n is independent of α so we conclude that it converges almost surely.

2. Gaussian multiplicative chaos

Gaussian multiplicative chaos is the theory introduced by Kahane [**43**] to generalize the idea of multiplicative cascade measures to a much more general situation, namely properly defining measures of the form

$$e^{\beta X(x)}\sigma(dx) \tag{3.16}$$

in situations where the covariance of the Gaussian process X may be very general, σ may be an arbitrary Radon measure and even the space X lives on might be a very general metric space. In this section, we mention some basic results reviewed in [**61**].

Let us formulate as a theorem an example of what one can say about such measures in a very general situation (see [**43, 61**])

THEOREM 3.4. *Let (D,ρ) be a locally compact metric space and let $K : D\times D \to [0,\infty]$ be of the form*

$$K(x,y)=\sum_{k=1}^{\infty}K_k(x,y), \tag{3.17}$$

where $K_k : D\times D\to[0,\infty)$ are continuous, non-negative and positive definite kernels of covariance operators. Then let $(Y_k)_k$ be a sequence of independent centered Gaussian processes on D such that Y_k has covariance K_k and let $X_n=\sum_{k=1}^n Y_k$. If σ is a Radon measure on D and $\beta\geq 0$, then the measure $M_{n,\beta}$ defined on Borel sets of D by

$$M_{n,\beta}(A)=\int_A e^{\beta X_n(x)-\frac{\beta^2}{2}\mathbb{E}(X_n(x)^2)}\sigma(dx) \tag{3.18}$$

converges weakly almost surely in the space of Radon measures on D to a random measure M_β. Moreover, the law of M_β is independent of the way we choose the sequence of covariances K_k - it only depends on K.

The convergence of the measure follows again from the fact that $M_{n,\beta}(A)$ is a non-negative martingale for each compact set A. We note that in our representation of the two-dimensional Gaussian free field, the covariance was written as a sum, but the summands were not non-negative so this approach can not be directly used. We remark that such an expansion still does exist for the GFF [**61**] so the corresponding measure for the GFF has an interpretation as a multiplicative chaos measure.

Proving that the limit is non-trivial requires some assumptions about the covariance and measure σ. While there are results in the general situation of a locally compact metric space and Radon measures, let us specialize to the situation where D is a domain in $\mathbb{R}^d$ and we consider the Lebesgue measure. In this setup, Kahane's theorem about the non-triviality about the limit measure becomes

THEOREM 3.5. *Let D be a domain in $\mathbb{R}^d$, $g : D\times D\to\mathbb{R}$ some bounded continuous non-negative function, $T>0$ and $K : D\times D\to[0,\infty]$*

$$K(x,y)=\log\frac{T}{|x-y|}+g(x,y) \tag{3.19}$$

be such that it can be written as a sum $K=\sum_k K_k$ as before. Then $M_{\beta,n}(A)$ is a uniformly integrable martingale and $M_\beta(A)$ is non-trivial for each compact set $A\subset D$ if and only if $\beta<\sqrt{2d}$.

Let us give a proof for this for $\beta^2 < d$. By the Gaussian property of the fields:

$$\begin{aligned}
\mathbb{E}(M_{n,\beta}^2(A)) &= \mathbb{E}\left(\int_{A\times A} dxdy e^{\beta(X_n(x)+X_n(y))-\frac{\beta^2}{2}\mathbb{E}(X_n(x)^2+X_n(y)^2)}\right) \\
&\leq \int_{A\times A} e^{\beta^2 K(x,y)}dxdy \\
&\leq C\int_{A\times A}|x-y|^{-\beta^2},
\end{aligned}$$

this is finite precisely when $d > \beta^2$. Thus from L^2 boundedness we have uniform integrability.

We note that it follows from our white noise decompositions that these results immediately give the existence of the 'high-temperature' measure for the fields constructed in terms of the functions g_i we considered (for them, the covariance has a natural representation as a sum coming from the time integral in the white noise representation). A corresponding result was proven recently independently for the two-dimensional Gaussian free field in the context of studying the KPZ-relation [**29**] (though in this approach, the regularization was done through taking averages of the field on circles which does not give a regularization of the type this theorem describes, but in [**61**] it is proved that the limit measures are equal in law).

There has been much research into such measures recently. In particular, in analogy to the multiplicative cascade case, the question of does there exist a normalization $f_{n,\beta}$ such that $f_{n,\beta}M_{\beta,n}$ converges for $\beta \geq \sqrt{2d}$ is a very natural one. At the critical point, this was answered in affirmative for a wide class of Gaussian fields in [**27, 28**]. Let us formulate their results as a theorem.

THEOREM 3.6. *Let X_t be a centered Gaussian process on $\mathbb{R}^d$ with covariance*

$$\mathbb{E}(X_t(x)X_t(y)) = \int_0^t k(e^s(x-y))ds \tag{3.20}$$

for some covariance kernel $k \in C^1(\mathbb{R}^d)$ with $k(0) = 1$ and compact support. Moreover, let $t \mapsto X_t$ have independent increments. Then

$$\sqrt{t}e^{\sqrt{2d}X_t(x)-dt}dx \tag{3.21}$$

converges in probability to a non-trivial limit. Moreover, this limiting measure can be described (up to a deterministic positive constant factor) by being the almost sure limit of

$$(\sqrt{2d}t - X_t(x))e^{\sqrt{2d}X_t(x)-dt}dx. \tag{3.22}$$

The limit measure has full support and is non-atomic.

In fact, the theorem can be proven for more general fields as well. For example, essentially the same arguments go through for the fields we considered in the previous chapter defined in terms of the white noise expansion with the functions g_1 and g_2 (g_2 does not fit the conditions stated here since its convolution square k is not smooth).

The situation with $\beta > \sqrt{2d}$ is still an open question currently, although in [**27**] it is remarked that with a normalization corresponding to the cascade situation, the sequence of measures is tight and every subsequential limit is non-trivial. The current belief is that these measures converge to a purely atomic measure exhibiting freezing behavior as for the cascade case: the critical measure determines the low temperature measures through some point process-type construction similar to the stable subordinator for the REM and multiplicative cascades.

Currently the strongest result in the $\beta > \sqrt{2d}$ case is for the discrete two-dimensional Gaussian free field in the $\beta = \infty$ situation [**19**]. Namely they prove that with the correct deterministic shift (as for the cascade case) the maximum of the discrete field converges in distribution to something non-trivial as we take the cutoff to zero (they consider the situation with fixed lattice spacing and take an infinite volume limit, but this is equivalent).

3. Some properties of the limit measure M_β

In this section we describe probabilistic properties (namely existence of moments) and geometric properties (namely modulus of continuity and Hausdorff dimension of the set the measure lives on) of the limit measure M_β in the Euclidean setup (with logarithmic singularity in the covariance). Corresponding results will also hold for multiplicative cascade measures. Our presentation continues to follow [**61**] closely.

From the point of view of probabilistic estimates of $M_\beta(A)$, the most natural thing is to see what kind of moments can be estimated. Kahane proved the following: for $\beta < \sqrt{2d}$, the measure M_β has finite positive moments of order p for $p \in (0, \frac{\sqrt{2d}}{\beta})$. In [**62**], it was shown that the measure also has negative moments. In [**27**], this result was extended to the critical case (for the type of fields considered in the paper). We collect these results into a theorem.

THEOREM 3.7. *For $\beta \leq \sqrt{2d}$ and any non-empty closed ball B, $\mathbb{E}(M_\beta(B)^p) < \infty$ if and only if $p < \frac{2d}{\beta^2}$.*

These moment results suggest that the tail of the distribution of $M_\beta(B)$, i.e. $\mathbb{P}(M_\beta(B) > x)$ is of the form $x^{-\frac{2d}{\beta^2}}$. This was indeed proven for the subcritical case in [**8**] and for the critical case in [**11**] for a one-dimensional field with covariance $\max(-\log|x-y|, 0)$ (namely the field constructed from the white noise expansion with the function g_1). We state this as a theorem as well

THEOREM 3.8. *For a centered Gaussian field on $\mathbb{R}$ with covariance of the form $\max(-\log|x-y|, 0)$, for any non-empty interval $I \subset \mathbb{R}$, there exists a constant $c = c_\beta$ such that for $\beta \leq \sqrt{2}$*

$$\lim_{x\to\infty} x^{\frac{2}{\beta^2}} \mathbb{P}(M_\beta(I) > x) = c. \tag{3.23}$$

From the point of view of geometry, once one knows that a measure is non-atomic, some of the first questions that come to mind are how does the measure of a ball depend on the radius and what size is the set the measure lives on. Kahane proved the following ([**43**])

THEOREM 3.9. *For $\beta < \sqrt{2d}$ we have that for a bounded domain D, there is a random constant $C = C_D$, which is almost surely finite such that we have for any ball $B(x,r)$ and any $\epsilon > 0$*

$$M_\beta(B(x,r)\cap D) \leq C r^{d-\frac{\beta^2}{2}-\epsilon}. \tag{3.24}$$

Moreover, almost surely there exists a subset $A \subset D$ such that A has full mass and the Hausdorff dimension of A is $d - \frac{\beta^2}{2}$.

A similar result was proven for the one-dimensional field considered above in [**11**]:

THEOREM 3.10. *For a centered Gaussian field on $\mathbb{R}$ with covariance of the form* $\max(-\log|x-y|, 0)$*, for any non-empty interval $I \subset [0,1]$, there exists a random constant C which is almost surely finite such that for any $\gamma < \frac{1}{2}$*

$$M_{\sqrt{2}}(I) \leq C\left(\log\left(1+|I|^{-1}\right)\right)^{-\gamma}. \tag{3.25}$$

Moreover, almost surely there exists a set $A \subset [0,1]$ such that $M_{\sqrt{2}}(A) = M_{\sqrt{2}}([0,1])$ and A has Hausdorff dimension zero.

4. Scaling and multifractal properties of multiplicative chaos

In this section, we consider some specific Gaussian multiplicative chaos measures that possess certain exact scaling properties. More precisely, we wish to consider fields for which the Gaussian multiplicative chaos measures satisfy a relation similar to (3.6).

In general, if the covariance is of the general Euclidean form considered in the previous sections, one can prove (see [**61**] for a sketch of the proof)

THEOREM 3.11. *For any $x \in D$ and $q \in [0, \frac{2d}{\beta^2})$,*

$$\mathbb{E}(M_\beta(B(x,r))^q) \sim r^{(d+\frac{\beta^2}{2})q - \frac{\beta^2}{2}q^2} \tag{3.26}$$

as $r \to 0$.

For this reason Gaussian multiplicative cascade measures are often called multifractal measures. If the measures had a single global characteristic fractal dimension, e.g. the measure of balls scaled like $M_\beta(B(x,r)) \sim r^\delta$, then the logarithm of the expectation would be linear in q. Thus when the logarithm is non-linear, it would seem that there is in some sense a local fractal behavior and this phenomenon is termed multifractality.

A natural question is then can one extend this type of behavior beyond moments into something e.g. distributional.

Let us begin with our regularized version of the Gaussian field on $\mathbb{R}$ with covariance $\max(-\log|x-y|, 0)$, namely the Gaussian field $X_t(x)$ given by

$$X_t(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} g_1(s,x,y)W(ds,dy), \tag{3.27}$$

where $g_1(s,x,y) = \mathbf{1}\{|y-x| \leq \frac{1}{2}\min(1, e^{-s}),\ s \leq t\}e^{\frac{s}{2}}$.

We now fix some interval $I \subset \mathbb{R}$ and assume at first that $I \subset [0,1]$ and $|I| < 1$. Write x_I for the midpoint of the interval and consider

$$g_I(s,y) = \mathbf{1}\left\{|y - x_I| \leq \frac{1}{2}\min(1-|I|, e^{-s}-|I|), s \leq -\log|I|\right\} e^{\frac{s}{2}}, \tag{3.28}$$

where $|I|$ is the length of I. Then for $t \geq -\log|I|$ and $x \in I$, $e^{-\frac{s}{2}}(g_1(s,x,y) - g_I(s,y)) \in \{0,1\}$ for all y and s. Let us thus write $g_1(s,x,y) = g_I(s,y) + h(s,x,y)$ and define the random variables

$$X(I) = \int_{\mathbb{R}}\int_{\mathbb{R}} g_I(s,y) W(ds,dy) \tag{3.29}$$

and

$$X_t^I(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} h(s,x,y) W(ds,dy), \tag{3.30}$$

where the proper definition is through the isonormal Gaussian process. We then note that $X(I)$ is independent of $X_t^I(x)$ for all $x \in I$ ($X(I)$ is the 'common part' of the field $X_t(x)$ for all the points $x \in I$). To deduce useful scaling properties, we want to determine what the field X_t^I looks like. Let us calculate its covariance: $X(I)$ being independent of $X_t^I(x)$ for all $x \in I$ we see that

$$\mathbb{E}(X_t^I(x)X_t^I(y)) = \mathbb{E}(X_t(x)X_t(y)) - \mathbb{E}(X(I)^2). \tag{3.31}$$

Now

$$\begin{aligned}
\mathbb{E}(X(I)^2) &= \int_{\mathbb{R}}\int_{\mathbb{R}} g_I(s,y)^2 ds dy \\
&= \int_{-\infty}^{-\log|I|} e^s \int_{x_I - \frac{1}{2}\min(1-|I|, e^{-s}-|I|)}^{x_I + \frac{1}{2}\min(1-|I|, e^{-s}-|I|)} dy ds \\
&= \int_{-\infty}^{-\log|I|} e^s \min(1-|I|, e^{-s}-|I|) ds \\
&= \int_{-\infty}^{0} e^s(1-|I|) ds + \int_0^{-\log|I|} (1 - e^s|I|) ds \\
&= -\log|I|.
\end{aligned}$$

Thus for $t \geq -\log|I|$ and $x,y \in I$

$$\begin{aligned}
\mathbb{E}(X_t^I(x)X_t^I(y)) &= \begin{cases} 1 + t - e^t|x-y| + \log|I|, & \text{for } |x-y| \leq e^{-t} \\ -\log|x-y| + \log|I|, & \text{for } e^{-t} \leq |x-y| \leq |I| \end{cases} \\
&= \begin{cases} 1 + (t+\log|I|) - e^{t+\log|I|}\frac{|x-y|}{|I|}, & \text{for } \frac{|x-y|}{|I|} \leq e^{-(t+\log|I|)} \\ -\log\frac{|x-y|}{|I|}, & \text{for } e^{-(t+\log|I|)} \leq \frac{|x-y|}{|I|} \leq 1 \end{cases}.
\end{aligned}$$

For $f(x) = \frac{1}{|I|}(x - x_I) + \frac{1}{2}$ we see that, $(X_t^I(x))_{x\in I} \overset{d}{=} (X_{t+\log|I|}(f(x)))_{x\in I}$. So for $\beta \leq \sqrt{2}$, plugging this decomposition of X_t into the definition of $M_{\beta,t}$ and passing to the $t \to \infty$, we see that

$$(M_\beta(A))_{A\in\mathcal{B}(I)} \overset{d}{=} \left(|I|e^{\beta X(I) - \frac{\beta^2}{2}\mathbb{E}(X(I)^2)}\tilde{M}_\beta(f(A))\right)_{A\in\mathcal{B}(I)}, \tag{3.32}$$

where $\mathcal{B}(I)$ is the family of Borel subsets of I. Here $\tilde{M}_\beta$ is a copy of M_β which is independent of $X(I)$. Note that as A runs through $\mathcal{B}(I)$, $f(A)$ runs through $\mathcal{B}([0,1])$. Specializing to a single set, we note that for any $A \subset [0,1]$ and $\lambda < 1$,

$$M_\beta(\lambda A) \overset{d}{=} \lambda^{1+\frac{\beta^2}{2}} e^{\beta X_\lambda} M_\beta(A), \tag{3.33}$$

where, X_λ is a centered Gaussian independent of $M_\beta(A)$ and of variance $-\log\lambda$. So indeed, this is an exact scaling relation that is much stronger than the moment relation of the previous theorem.

Finally we note that we could do this construction on any collection of disjoint intervals I simultaneously, but we would not get any independence between different intervals.

Let us now consider the field defined in terms of the function $g_2(s,x,y) = \mathbf{1}\{|x-y| \leq \frac{1}{2}e^{-s}, s \in [0,t]\}e^{\frac{s}{2}}$. Let $t \geq s \geq 0$ and consider $Y_{t,s}(x) = X_t(x) - X_s(x)$. $Y_{t,s}(x)$ is independent of $X_s(x)$ so we find for $x, y \in [0,1]$

$$\mathbb{E}(Y_{t,s}(x)Y_{t,s}(y)) = \begin{cases} t - s - (e^{t-s} - 1)e^s|x-y|, & \text{for } |x-y| \leq e^{-t} \\ -\log(e^s|x-y|) + e^s|x-y| - 1, & \text{for } e^{-t} \leq |x-y| \leq e^{-s} \\ 0, & \text{for } e^{-s} \leq |x-y| \end{cases}.$$

We conclude that $(X_t(x) - X_s(x))_{x\in[0,1]} \overset{d}{=} (X_{t-s}(e^s x))_{x\in[0,1]}$. Plugging this into the definition of M_β, keeping s fixed and taking $t \to \infty$, we find for this field,

$$(M_\beta(A))_{A\in\mathcal{B}([0,1])} \overset{d}{=} \left(e^{-s}\int_A e^{\beta X_s(x) - \frac{\beta^2}{2}\mathbb{E}(X_s(x)^2)} M_\beta^s(dx)\right)_{A\in\mathcal{B}([0,1])}, \tag{3.34}$$

where M_β^s is independent of X_s and $(M_\beta^s(A))_{A\in\mathcal{B}([0,1])} \overset{d}{=} (M_\beta(e^s A))_{A\in\mathcal{B}([0,1])}$.

So we have given examples of two specific cases of Gaussian multiplicative chaos measures which in slightly different ways generalize the notion of exact scaling that multiplicative cascade measures exhibit. We end our treatment about Gaussian multiplicative chaos with the following theorem combining results from [**27, 28, 5**]:

THEOREM 3.12. *Under the conditions of Theorem 3.6, the limit measure M_β for $\beta \leq \sqrt{2d}$ satisfies*

$$(M_\beta(A))_{A\in\mathcal{B}(\mathbb{R}^d)} \overset{d}{=} \left(e^{-ds}\int_A e^{\beta X_s(x) - \frac{\beta^2}{2}\mathbb{E}(X_s(x)^2)} M_\beta^s(dx)\right)_{A\in\mathcal{B}(\mathbb{R}^d)} \tag{3.35}$$

where M_β^s is independent of X_s and $(M_\beta^s(A))_{A\in\mathcal{B}(\mathbb{R}^d)} \overset{d}{=} (M_\beta(e^s A))_{A\in\mathcal{B}(\mathbb{R}^d)}$.

CHAPTER 4

Outline of the generating function approach

In this section, we give a very brief review of the ideas in [**68**] used to prove convergence of the total mass of the Gaussian multiplicative cascade measures. The approach follows [**23**] by noting that a generating function of the total mass $W_{n,\beta}$ (see the section on multiplicative cascades for the definition) satisfies a certain recursion relation. The hierarchical structure of the branching random walk implies that we can write $W_{n+1,\beta} \stackrel{d}{=} \frac{1}{2}e^{\beta y-\frac{\beta^2}{2}}(W^1_{n,\beta}+W^2_{n,\beta})$, where $W^1_{n,\beta}$ and $W^2_{n,\beta}$ are independent copies of $W_{n,\beta}$ which are independent of each other and y is a standard Gaussian independent of $W^1_{n,\beta}$ and $W^2_{n,\beta}$. Thus by independence

$$
\begin{aligned}
H_{n+1,\beta}(x) &= \mathbb{E}(\exp(-e^{\beta x}W_{n+1,\beta})) \\
&= \mathbb{E}(\exp(-e^{\beta(x+y-c(\beta))}W^1_{n,\beta})\exp(-e^{\beta(x+y-c(\beta))}W^1_{n,\beta})) \\
&= \mathbb{E}(H_{n,\beta}(x+y-c(\beta))^2) \\
&= \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} H_{n,\beta}(x+y-c(\beta))^2 dy,
\end{aligned}
\tag{4.1}
$$

where $c(\beta)=\frac{\beta}{2}+\frac{\log 2}{\beta}$ and $H_{0,\beta}(x)=\exp(-e^{\beta x})$. To have a similar notation as in [**68**], we write $G_{n,\beta}(x)=H_{n,\beta}(-x+c(\beta)n)$. Then $G_{n,\beta}$ satisfies the recursion

$$
G_{n+1,\beta}(x) = \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} G_{n,\beta}(x+y)^2 dy. \tag{4.2}
$$

The main result of [**68**] is the following

THEOREM 4.1. *Let $m_{n,\beta}=G^{-1}_{n,\beta}(\frac{1}{2})$. Then $G_{n,\beta}(x+m_{n,\beta})$ converges to the unique increasing function $w_\beta:\mathbb{R}\to[0,1]$ satisfying $w_\beta(-\infty)=0$, $w_\beta(\infty)=1$, $w_\beta(0)=\frac{1}{2}$ and*

$$
w_\beta(x) = \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} w_\beta(x+y+\tilde{c}(\beta))^2 dy, \tag{4.3}
$$

where $\tilde{c}(\beta)=c(\beta)$ for $\beta\leq\sqrt{2\log 2}$ and $\tilde{c}(\beta)=c(\sqrt{2\log 2})$ for $\beta>\sqrt{2\log 2}$. Moreover,

$$
m_{\beta,n} = \begin{cases} c(\beta)n+a, & \text{for } \beta<\sqrt{2\log 2} \\ \sqrt{2\log 2}n-\frac{1}{2\sqrt{2\log 2}}\log n+\mathcal{O}(1), & \text{for } \beta=\sqrt{2\log 2} \\ \sqrt{2\log 2}n-\frac{3}{2\sqrt{2\log 2}}\log n+\mathcal{O}(1), & \text{for } \beta>\sqrt{2\log 2} \end{cases}, \tag{4.4}
$$

for some constant a.

These results imply for example that up to a deterministic factor which is bounded and bounded away from zero, $\sqrt{n}W_{n,\beta}$ converges in law. Similarly, this gives the normalization of the 'low-temperature' measure up to a deterministic factor which is bounded and bounded away from zero.

We point out the freezing transition occurring at $\beta_c = \sqrt{2\log 2}$: $\tilde{c}(\beta)$ and thus w_β become independent of β. We also mention the relationship between the freezing transition and the stable subordinator: let W_β be the limit of $W_{n,\beta}$ after the correct normalization. The previous theorem implies that $\mathbb{E}(\exp(-e^{-\beta x}W_\beta)) = \mathbb{E}(\exp(-e^{-\beta_c x}W_{\beta_c}))$ for all x. On the other hand, if $L_{\frac{\beta_c}{\beta}}$ is a stable subordinator of index $\frac{\beta_c}{\beta}$ (so that $\mathbb{E}(\exp(-sL_\alpha(t))) = e^{-ts^\alpha}$) independent of W_{β_c} then $\mathbb{E}(\exp(-e^{-\beta x}L_{\frac{\beta_c}{\beta}}(W_{\beta_c}))) = \mathbb{E}(\exp(-e^{-\beta_c x}W_{\beta_c})) = \mathbb{E}(\exp(-e^{-\beta x}W_\beta))$ so we see that $L_{\frac{\beta_c}{\beta}}(W_{\beta_c}) \stackrel{d}{=} W_\beta$. This phenomenon should thus occur in any model where the generating function $\mathbb{E}(\exp(-e^{-\beta x}W_\beta))$ becomes independent of the temperature.

The analysis of the recursion (4.2) is based on the remark that it is a discrete time version of the so called KPP-equation. If one considered a branching Brownian motion instead of a branching random walk, one would end up with the equation

$$\partial_t G_t(x) = \frac{1}{2}\partial_x^2 G_t(x) + G_t(x)^2 - G_t(x). \tag{4.5}$$

This is an equation analyzed in great detail in [**18**]. In particular, similar convergence results are proven. The philosophy of [**68**] is to follow [**18**] as closely as possible and cut corners when possible.

In fact the proof of convergence of $G_{n,\beta}(x + m_{n,\beta})$ with the implicit definition of $m_{n,\beta}$ can be done in quite a simple manner. For $\beta < \sqrt{2\log 2}$, it is essentially just a martingale argument. The $\beta \geq \sqrt{2\log 2}$ case is a bit more complicated. A central tool for it is the following 'maximum principle'

THEOREM 4.2. *Let G_n^1 and G_n^2 be given by the recursion* (4.2) *with initial data G_0^1 and G_0^2 with the property that $G_0^2(x) > G_0^1(x)$ for $x > x_0$ and $G_0^2(x) < G_0^1(x)$ for $x < x_0$. Then there is a point $x_n \in [-\infty, \infty]$ such that $G_n^2(x) > G_n^1(x)$ for $x > x_n$ and $G_n^2(x) < G_n^1(x)$ for $x < x_n$. Moreover, if $|x_n| = \infty$ for some n, then $x_m = x_n$ for $m \geq n$.*

Using this result, one can show for example that for $x \geq 0$, $G_{n,\beta}(x + m_{n,\beta})$ is increasing in β. Also with a simple application of this result and some elementary analysis, one can show that $G_{n,\infty}(x + m_{n,\infty})$ converges to $w_{\sqrt{2\log 2}}$. Morally, one then has for $x \geq 0$ and $\beta \geq \sqrt{2\log 2}$, by the 'high-temperature' convergence that $w_{\sqrt{2\log 2}-\delta}(x) - \epsilon \leq G_{n,\beta}(x + m_{n,\beta}) \leq w_{\sqrt{2\log 2}}(x) + \epsilon$. Then taking $\delta \to 0$ gives the result.

The more technical (and less self contained) part of [**68**] is analyzing $m_{n\beta}$. The starting point for this, is (as in [**18**]) a Feynman-Kac representation of the generating function G. Iterating the recursion and writing $U = 1 - G$, one obtains

$$U_{n,\beta}(x) = \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{2n}}}{\sqrt{2\pi n}} U_{0,\beta}(y)\mathbb{E}_n^{x,y}\left(e^{\sum_{m=1}^n k_{n-m}(Y_m)}\right) dy, \tag{4.6}$$

where $k_m = \log(1 + G_{m,\beta})$, Y is a discrete time Brownian bridge and $\mathbb{E}_n^{x,y}$ is the expectation of a discrete time Brownian bridge over $\{0, ..., n\}$ from x to y.

We will give a very brief heuristic picture of what is going on in the analysis of this representation. The moral is that we wish to balance the two factors - the Gaussian part and the Brownian bridge bart. For the Gaussian part, we know that since $G_{n,\beta}(x + m_{n,\beta}) \to w_{\sqrt{2\log 2}}$, the leading part to $m_{n,\beta}$ is $\sqrt{2\log 2}n$. Thus the leading part of the Gaussian will always be 2^{-n}. For the Brownian bridge part, we note that $k_m(x) \leq \log 2$ and $k_l(x) \approx \log 2$ when $x = m_{l,\beta} + C$ where C is big. So the Brownian bridge part can cancel the smallness of the Gaussian part only for paths that are well above the curve $\frac{n-s}{n}x + \frac{s}{n}y + m_{n-s,\beta}$. The technical analysis then goes into making this picture rigorous and showing that in fact

$$\mathbb{E}_n^{x+m_n,y}\left(e^{\sum_{m=1}^n k_{n-m}(Y_m)}\right) \approx 2^n \mathbb{P}_n^{x,y}\left(Y_k \geq 0, \text{ for } k \leq n\right) \approx 2^n \frac{xy}{n}. \tag{4.7}$$

We finally end up with the problem of determining the lower order behavior of $m_{n,\beta} = \sqrt{2\log 2}n + \epsilon_n$ (where we assume that $\frac{\epsilon_n^2}{n} \to 0$ as $n \to \infty$): we want the following integral (approximating $G_{n,\beta}(\mathcal{O}(1) + m_{n,\beta})$) to be bounded and bounded away from zero

$$\int_1^\infty \frac{e^{-\frac{(\sqrt{2\log 2}n + \epsilon_n - y)^2}{2n}}}{\sqrt{2\pi n}} U_{0,\beta}(y) 2^n \frac{y}{n} dy, \tag{4.8}$$

where $U_{0,\beta}(y) \sim e^{-\beta y}$ as $y \to \infty$. This is a simple problem and one finds $\epsilon_n = -\frac{1}{2\sqrt{2\log 2}}\log n + \mathcal{O}(1)$ for $\beta = \sqrt{2\log 2}$ and $\epsilon_n = -\frac{3}{2\sqrt{2\log 2}}\log n + \mathcal{O}(1)$ for $\beta > \sqrt{2\log 2}$.

There is a lot of technical work in making this precise. This approach is also not that well suited for generalization to the multiplicative chaos case - the recursions become non-local in the parameter n in general. A Feynman-Kac type of representation should be possible in the multiplicative chaos situation, but at least so far, other methods have been more successful. Due to these facts and that [**68**] is not that self contained, we will give alternative approaches to the proof of what $m_{n,\beta}$ looks like in the following chapters.

We close this section with a comment about the branching random walk we called Y. Let us denote the corresponding quantities for this branching random walk with a bar. Recalling the definition of this branching random walk, we see that the total mass $\bar{W}_{n,\beta}$ satisfies the recursion $\bar{W}_{n+1,\beta} \stackrel{d}{=} \frac{1}{2}e^{\beta y_1 - \frac{\beta^2}{2}}\bar{W}^1_{n,\beta} + \frac{1}{2}e^{\beta y_2 - \frac{\beta^2}{2}}\bar{W}^2_{n,\beta}$. In terms of generating functions, this recursion becomes

$$\bar{G}_{n+1,\beta}(x) = \left(\int_{\mathbb{R}} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \bar{G}_{n,\beta}(x+y)dy\right)^2 \tag{4.9}$$

with initial data $\bar{G}_{0,\beta}(x) = \exp(-e^{-\beta x})$. Now if we had started with such a recursion and defined $\tilde{G}_{n,\beta} = \sqrt{\bar{G}_{n,\beta}}$, then $\tilde{G}_{n,\beta}$ would satisfy the same recursion as $G_{n,\beta}$ with initial data $\tilde{G}_{0,\beta}(x) = G_{0,\beta}(x + \frac{\log 2}{\beta})$. The recursions we consider are invariant under translations in the spatial coordinate so we see that studying convergence questions for $\bar{G}$ and G are equivalent so the question of the correct normalization and convergence of the multiplicative cascade measure for Y is equivalent to that of X.

CHAPTER 5

The critical normalization

In this section, we shall show that $m_{n,\sqrt{2\log 2}} = \sqrt{2\log 2}n - \frac{1}{2\sqrt{2\log 2}}\log n + \mathcal{O}(1)$ through branching random walk theory based on an argument in [**3**]. We will see that our argument will not depend on the branching structure in any way, but will carry through for the multiplicative chaos situation as well (given the knowledge that the derivative martingale converges). Namely in this case, it will say that $\log(\sqrt{n}W_{n,\sqrt{2\log 2}})$ is tight. Proving that $\sqrt{n}W_{n,\sqrt{2\log 2}}$ converges with these methods, would require more work and we refer to [**3**] for the details of this.

1. A change of measure

Our proof is centered around the derivative martingale and its truncated form. In this chapter, we will write $W_n = W_{n,\sqrt{2\log 2}}$. For $\alpha \geq 0$, let us define the quantities

$$W_n^{(\alpha)} = \sum_{\sigma\in\Sigma_n} e^{\sqrt{2\log 2}(X_\sigma - \sqrt{2\log 2}n)}\mathbf{1}\{X_{\sigma|k} - \sqrt{2\log 2}k \leq \alpha, \text{ for } k \leq n\} \tag{5.1}$$

and

$$\begin{aligned} D_n^{(\alpha)} = \sum_{\sigma\in\Sigma_n} & R(\sqrt{2\log 2}n - X_\sigma + \alpha)e^{\sqrt{2\log 2}(X_\sigma - \sqrt{2\log 2}n)} \\ & \times \mathbf{1}\{X_{\sigma|k} - \sqrt{2\log 2}k \leq \alpha, \text{ for } k \leq n\}, \end{aligned} \tag{5.2}$$

where R is the renewal function of the random walk with standard Gaussian increments. We recall that $D_n^{(\alpha)}$ is a positive martingale. Let $\mathcal{F}_n = \sigma(X_\sigma : \sigma \in \cup_{k\leq n}\Sigma_k)$. Since we are dealing with a positive martingale, we can define a probability measure $Q^{(\alpha)}$ with the property that $dQ^{(\alpha)}|\mathcal{F}_n = \frac{D_n^{(\alpha)}}{R(\alpha)}d\mathbb{P}|\mathcal{F}_n$. Our philosophy is to first prove that under $Q^{(\alpha)}$, $\log\frac{\sqrt{n}W_n^{(\alpha)}}{D_n^{(\alpha)}}$ is tight. We then prove that for large enough α, $W_n^{(\alpha)} = W_n$ and $D_n^{(\alpha)} \approx D_n$ and argue that this implies that $\log(\sqrt{n}W_n)$ is tight under $\mathbb{P}$.

Proposition 5.1. *For a fixed $\alpha > 0$, $\log\frac{\sqrt{n}W_n^{(\alpha)}}{D_n^{(\alpha)}}$ is tight with respect to the measure $Q^{(\alpha)}$.*

Proof. We make use of the result that a sequence of random variables A_n is tight if and only if for any deterministic sequence c_n such that $c_n \to 0$, c_nA_n converges to zero in probability [**45**]. Thus we wish to show that for any such sequence and any $\epsilon > 0$,

$$Q^{(\alpha)}\left(|c_n|\left|\log\frac{\sqrt{n}W_n^{(\alpha)}}{D_n^{(\alpha)}}\right| > \epsilon\right) \to 0 \tag{5.3}$$

as $n \to \infty$.

We shall do this by estimating $\mathbb{E}_{Q^{(\alpha)}}(\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}})$ and $\mathbb{E}_{Q^{(\alpha)}}(\frac{D_n^{(\alpha)}}{W_n^{(\alpha)}})$. By Theorem 2.1 and Proposition 2.3

$$\begin{aligned}
\mathbb{E}_{Q^{(\alpha)}}\left(\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\right) &= \frac{1}{R(\alpha)}\mathbb{E}_{\mathbb{P}}\left(W_n^{(\alpha)}\right) \\
&= \frac{1}{R(\alpha)}\mathbb{P}(S_k \le \alpha, \text{ for } k \le n) \\
&\sim n^{-\frac{1}{2}}.
\end{aligned}$$

We note that by Jensen's inequality

$$\begin{aligned}
\left(\frac{D_n^{(\alpha)}}{W_n^{(\alpha)}}\right)^2 \le \frac{1}{W_n^{(\alpha)}} \sum_{\sigma \in \Sigma_n} & R(\sqrt{2\log 2}n - X_\sigma + \alpha)^2 e^{\sqrt{2\log 2}(X_\sigma - \sqrt{2\log 2}n)} \\
& \times \mathbf{1}\{X_{\sigma|k} - \sqrt{2\log 2}k \le \alpha \text{ for } k \le n\}.
\end{aligned}$$

Thus by Theorem 2.1

$$\begin{aligned}
&\mathbb{E}_{Q^{(\alpha)}}\left(\frac{D_n^{(\alpha)}}{W_n^{(\alpha)}}\right) = \frac{1}{R(\alpha)}\mathbb{E}_{\mathbb{P}}\left(W_n^{(\alpha)}\left(\frac{D_n^{(\alpha)}}{W_n^{(\alpha)}}\right)^2\right) \\
&\le \frac{1}{R(\alpha)}\mathbb{E}\Bigg(\sum_{\sigma \in \Sigma_n} R(\sqrt{2\log 2}n - X_\sigma + \alpha)^2 e^{\sqrt{2\log 2}(X_\sigma - \sqrt{2\log 2}n)} \\
&\qquad \times \mathbf{1}\{X_{\sigma|k} - \sqrt{2\log 2}k \le \alpha \text{ for } k \le n\}\Bigg) \\
&= \frac{1}{R(\alpha)}\mathbb{E}(R(\alpha - S_n)^2 \mathbf{1}\{S_k \le \alpha, \text{for } k \le n\}).
\end{aligned}$$

Recall that by the renewal theorem (Theorem 2.4), $R(x) \sim x$ as $x \to \infty$. Thus by Proposition 2.3

$$
\begin{aligned}
&\mathbb{E}_{Q^{(\alpha)}}\left(\frac{D_n^{(\alpha)}}{W_n^{(\alpha)}}\right)\\
&\le \frac{1}{R(\alpha)}\int_{-\infty}^{\alpha} dy\frac{e^{-\frac{y^2}{2n}}}{\sqrt{2\pi n}}R(\alpha-y)^2\mathbb{P}_n^{0,y}(Y_k\le\alpha,\ \text{for } k\le n)\\
&\le \frac{1}{R(\alpha)}\int_{-\infty}^{0} dy\frac{e^{-\frac{(y+\alpha)^2}{2n}}}{\sqrt{2\pi n}}R(-y)^2\mathbb{P}_n^{-\alpha,y}(Y_k\le 0,\ \text{for } k\le n)\\
&\le \frac{1}{R(\alpha)}\int_{-\infty}^{0} dy\frac{e^{-\frac{(y+\alpha)^2}{2n}}}{\sqrt{2\pi n}}C(1-y)^2\frac{(1+\alpha)(1-y)}{n}\\
&\le C\frac{1+\alpha}{R(\alpha)}\int_{0}^{\infty} dy\frac{e^{-\frac{(y-\frac{\alpha}{\sqrt{n}})^2}{2}}}{\sqrt{2\pi}}(1+\sqrt{n}y)^3n^{-1}\\
&\le C\sqrt{n}\int_{0}^{\infty} dy\frac{e^{-\frac{(y-\frac{\alpha}{\sqrt{n}})^2}{2}}}{\sqrt{2\pi}}(1+y)^3\\
&\le C\sqrt{n}.
\end{aligned}
$$

By Markov's inequality

$$
\begin{aligned}
Q^{(\alpha)}\left(|c_n|\left|\log\frac{\sqrt{n}W_n^{(\alpha)}}{D_n^{(\alpha)}}\right|>\epsilon\right) &\le Q^{(\alpha)}\left(\frac{\sqrt{n}W_n^{(\alpha)}}{D_n^{(\alpha)}}>e^{\frac{\epsilon}{|c_n|}}\right)\\
&\quad + Q^{(\alpha)}\left(\frac{D_n^{(\alpha)}}{\sqrt{n}W_n^{(\alpha)}}>e^{\frac{\epsilon}{|c_n|}}\right)\\
&\le Ce^{-\frac{\epsilon}{|c_n|}}
\end{aligned}
$$

which converges to zero as $n\to\infty$. □

2. Tightness under the original measure

We will now show that Proposition 5.1 implies our desired result, i.e. that

THEOREM 5.2. $\log(\sqrt{n}W_n)$ *is tight under the measure* $\mathbb{P}$.

PROOF. We use the same characterization of tightness as in Proposition 5.1. Let c_n be a sequence such that $c_n\to 0$ and let $\epsilon>0$. Our goal is to show that

$$
\lim_{n\to\infty}\mathbb{P}(|c_n\log\sqrt{n}W_n|>\epsilon)=0. \tag{5.4}
$$

Our starting point is Proposition 5.1. Applied to our choice of c_n and ϵ, it says that for any fixed $\alpha\ge 0$,

$$
\mathbb{E}_{\mathbb{P}}\left(D_n^{(\alpha)}\mathbf{1}\left\{\left|c_n\log\sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\right|>\epsilon\right\}\right)\to 0. \tag{5.5}
$$

We note that $\max_{\sigma\in\Sigma_n}X_\sigma-\sqrt{2\log 2}n\to-\infty$ a.s. (since $W_n\to 0$ a.s.) so if we define $\Omega_k=\{\sup_{\sigma\in\cup_k\Sigma_k}(X_\sigma-\sqrt{2\log 2}|\sigma|)\le k\}$, then Ω_k is an increasing sequence of sets and $\mathbb{P}(\Omega_k) \to 1$ as $k \to \infty$. Let $\eta > 0$ and k_0 be such that $\mathbb{P}(\Omega_{k_0}) \geq 1 - \eta$. Restricting to Ω_{k_0}, we have (by positivity)

$$\mathbb{E}\left(D_n^{(\alpha)} \mathbf{1}\left\{ \left| c_n \log \sqrt{n} \frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} \right| > \epsilon \right\} \mathbf{1}_{\Omega_{k_0}} \right) \to 0. \tag{5.6}$$

Let us now consider $D_n^{(\alpha)}$ and $W_n^{(\alpha)}$ more carefully. If $\alpha \geq k_0$, then on Ω_{k_0}, $W_n^{(\alpha)} = W_n$. To estimate $D_n^{(\alpha)}$, we note that since by the renewal theorem (Theorem 2.4) there is a c_0 so that $\frac{R(x)}{x} \to c_0$ as $x \to \infty$, there is a constant $M = M(\epsilon) > 0$ so that for $x \geq M$,

$$c_0(1-\epsilon)x \leq R(x) \leq c_0(1+\epsilon)x. \tag{5.7}$$

Let us now fix $\alpha = k_0 + M$. For this choice of α we have on Ω_{k_0}

$$\begin{aligned} 0 &< c_0(1-\epsilon)(\sqrt{2\log 2}|\sigma| - X_\sigma + \alpha) \\ &\leq R(\sqrt{2\log 2}|\sigma| - X_\sigma + \alpha) \\ &\leq c_0(1+\epsilon)(\sqrt{2\log 2}|\sigma| - X_\sigma + \alpha). \end{aligned}$$

By our choice of α, $\mathbf{1}\{X_{\sigma|k} - \sqrt{2\log 2}k \leq \alpha\} = 1$ for all $\sigma \in \Sigma_n$ on Ω_{k_0} so multiplying the inequalities by $e^{\sqrt{2\log 2}(X_\sigma - \sqrt{2\log 2}n)}$ and summing over $\sigma \in \Sigma_n$, we get

$$c_0(1-\epsilon)(D_n + \alpha W_n) \leq D_n^{(\alpha)} \leq c_0(1+\epsilon)(D_n + \alpha W_n). \tag{5.8}$$

We know that $D_n \to D > 0$ almost surely and $W_n \to 0$ almost surely. We conclude that on Ω_{k_0}, almost surely $\liminf_{n\to\infty} D_n^{(\alpha)} \geq (1-\epsilon)c_0 D > 0$. Thus (5.6) implies that

$$\mathbf{1}\left\{ \left| c_n \log \frac{\sqrt{n} W_n}{D_n^{(\alpha)}} \right| > \epsilon \right\} \mathbf{1}_{\Omega_{k_0}} \tag{5.9}$$

converges to zero in probability (this follows from $D_n^{(\alpha)}$ being almost surely bounded from below by a positive random variable and $L^1(\mathbb{P})$ convergence implying convergence in probability).

Let us define $K_n = \mathbf{1}_{\Omega_{k_0}} c_n \log(\sqrt{n} W_n)$ and $L_n = \mathbf{1}_{\Omega_{k_0}} c_n \log D_n^{(\alpha)}$. So in terms of these random variables, our convergence in probability implies that $K_n - L_n$ converges to zero in probability. It follows from (5.8) and the fact that D_n converges almost surely to D which is almost surely finite, that $c_n \log D_n^{(\alpha)}$ converges almost surely to zero. Since $K_n - L_n$ converges in probability and L_n converges almost surely, we see that K_n converges to zero in probability, i.e.

$$\mathbb{P}(\Omega_{k_0} \cap \{|c_n \log(\sqrt{n} W_n)| > \epsilon\}) \to 0. \tag{5.10}$$

Since $\mathbb{P}(\Omega_{k_0}) \geq 1 - \eta$, this implies that

$$\limsup_{n\to\infty} \mathbb{P}(|c_n \log(\sqrt{n}W_n)| > \epsilon) \leq \eta. \tag{5.11}$$

As $\eta > 0$ was arbitrary, this implies that $c_n \log(\sqrt{n}W_n)$ converges to zero in probability and $\log(\sqrt{n}W_n)$ is tight. □

3. The Gaussian multiplicative chaos situation

Note that apart from the information concerning the derivative martingale, nothing in our argument depended on the correlations of the field. We simply used a many-to-one argument and Jensen's inequality. Thus our argument carries through in case where we have discretized a logarithmically correlated field as well. Also with minor notational modifications, these arguments work for white noise expansions as well (as long as one knows convergence of the derivative martingale and the positivity of its limit).

CHAPTER 6

The supercritical normalization

In this chapter, we will prove that for $\beta > \sqrt{2\log 2}$, $m_{n,\beta} = \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + \mathcal{O}(1)$. We will do this by showing that there exist constants $C_1, C_2 > 0$ so that for $n \geq 1$ and $x \in [1, \log n]$,

(6.1)
$$C_1 e^{-\sqrt{2\log 2}x} \leq \mathbb{P}\left(\frac{1}{\beta}\log\left(n^{\frac{3\beta}{2\sqrt{2\log 2}}} e^{\frac{1}{2}(\beta-\sqrt{2\log 2})^2 n} W_{n,\beta}\right) \geq x\right) \leq C_2 x e^{-\sqrt{2\log 2}x}.$$

If this were not the correct form for $m_{n,\beta}$, this probability would converge to either zero or one.

Our approach is based on [**52**]. The essential idea is that the maximum of the branching random walk controls the low-temperature regime so the proof is very similar to proving the corresponding result for the maximum. Indeed, we will need the corresponding result for the maximum. Again, much of this will go through for more general correlations. We will split our proof into three parts.

Proposition 6.1. *There exists a constant $C > 0$ so that for $n \geq 1$ and $x \in [1, \log n]$*

$$\mathbb{P}\left(\max_{\sigma\in\Sigma_n} X_\sigma \geq \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x\right) \geq C e^{-\sqrt{2\log 2}x}. \tag{6.2}$$

Note that as $\log \sum_\sigma e^{\beta X_\sigma} \geq \beta \max_\sigma X_\sigma$, the lower bound on the tail of the maximum implies the lower bound on the tail of $\log W_{n,\beta}$ we desire. While it is not important for tightness, we note that the lower bound is not sharp. In fact the asymptotic behavior of the tail is $Cxe^{-\sqrt{2\log 2}x}$, but the previous results are enough for us.

Proposition 6.2. *There exists a constant $C > 0$ so that for $n \geq 1$ and $x \in [1, \log n]$*

$$\mathbb{P}\left(\max_{\sigma\in\Sigma_n} X_\sigma \geq \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x\right) \leq C x e^{-\sqrt{2\log 2}x}. \tag{6.3}$$

Proposition 6.3. *There exists a constant $C > 0$ so that for $n \geq 1$ and $x \in [1, \log n]$,*

$$\mathbb{P}\left(\frac{1}{\beta}\log\left(n^{\frac{3\beta}{2\sqrt{2\log 2}}} e^{\frac{1}{2}(\beta-\sqrt{2\log 2})^2 n} W_{n,\beta}\right) \geq x\right) \leq C_2 x e^{-\sqrt{2\log 2}x} \tag{6.4}$$

1. Proof of the lower bound for the tail of the maximum

Our argument will follow [**69**]. The proof is a fairly simple second moment estimate applied to a random variable that keeps track of the behavior of the branching random walk at all times. Define

$$Y_n = \sum_{\sigma\in\Sigma_n} \mathbf{1}\Big\{X_\sigma - \sqrt{2\log 2}n + \frac{3}{2\sqrt{2\log 2}}\log n \in [x-1,x];$$
$$X_{\sigma|k} \le \frac{k}{n}\left(\sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x\right) \text{ for all } k\le n\Big\}.$$

We then note that

$$\begin{aligned}
&\mathbb{P}\left(\max_{\sigma\in\Sigma_n} X_\sigma \ge \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x - 1\right)\\
&= \mathbb{P}\left(X_\sigma \ge \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x - 1, \text{ for some } \sigma\in\Sigma_n\right)\\
&\ge \mathbb{P}(Y_n\ge 1).
\end{aligned}$$

We note that by Cauchy-Schwarz, for a random variable with non-negative integer values

$$\mathbb{E}(Y_n) = \mathbb{E}(Y_n; Y_n\ge 1) \le \sqrt{\mathbb{E}(Y_n^2)}\sqrt{\mathbb{P}(Y_n\ge 1)}. \tag{6.5}$$

Thus

$$\mathbb{P}\left(\max_{\sigma\in\Sigma_n} X_\sigma \ge \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x\right) \ge \frac{(\mathbb{E}(Y_n))^2}{\mathbb{E}((Y_n)^2)}. \tag{6.6}$$

By Theorem 2.1, Proposition 2.3 and an elementary property of the Brownian bridge (along with recalling that $x\in[1,\log n]$)

$$\begin{aligned}
\mathbb{E}(Y_n) = \mathbb{E}\bigg(&e^{-\sqrt{2\log 2}S_n}\mathbf{1}\Big\{S_n + \frac{3}{2\sqrt{2\log 2}}\log n \in [x-1,x],\\
&S_k \le -\frac{3k}{2n\sqrt{2\log 2}}\log n + \frac{k}{n}x \text{ for all } k\le n\Big\}\bigg)\\
&\asymp n^{\frac{3}{2}}e^{-\sqrt{2\log 2}x}\int_{x-1-\frac{3}{2\sqrt{2\log 2}}\log n}^{x-\frac{3}{2\sqrt{2\log 2}}\log n} dz\frac{e^{-\frac{z^2}{2n}}}{\sqrt{2\pi n}}\\
&\mathbb{P}_n^{0,z}\left(Y_k \le -\frac{3k}{2n\sqrt{2\log 2}}\log n + \frac{k}{n}x \text{ for all } k\le n\right)\\
&= n^{\frac{3}{2}}e^{-\sqrt{2\log 2}x}\int_{x-1-\frac{3}{2\sqrt{2\log 2}}\log n}^{x-\frac{3}{2\sqrt{2\log 2}}\log n} \frac{e^{-\frac{z^2}{2n}}}{\sqrt{2\pi n}}\\
&\mathbb{P}_n^{0,z+\frac{3}{2\sqrt{2\log 2}}\log n - x}(Y_k\le 0 \text{ for all } k\le n)\\
&\asymp Ce^{-\sqrt{2\log 2}x}\int_0^1 e^{-\frac{(z+x-\frac{3}{2\sqrt{2\log 2}}\log n)^2}{2n}}dz\\
&\asymp Ce^{-\sqrt{2\log 2}x}.
\end{aligned}$$

Here $a_n \asymp b_n$ means that $\frac{a_n}{b_n}$ is uniformly bounded and uniformly bounded away from zero.

We shall use Theorem 2.2 to estimate $\mathbb{E}(Y_n^2)$: let

$$Y_{n,\sigma} = \mathbf{1}\Big\{X_\sigma - \sqrt{2\log 2}n + \frac{3}{2\sqrt{2\log 2}}\log n \in [x-1,x];$$
$$X_{\sigma|k} \leq \frac{k}{n}\left(\sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x\right) \text{ for all } k \leq n\Big\}.$$

Then by Theorem 2.2:

$$\begin{aligned}\mathbb{E}\left((Y_n)^2\right) &= \sum_{\sigma,\sigma'\in\Sigma_n} \mathbb{E}(Y_{n,\sigma}Y_{n,\sigma'})\\ &= \sum_{l=1}^{n} \mathbb{E}\Big(e^{-\sqrt{2\log 2}S_n} e^{-\sqrt{2\log 2}S'_{n-l}} \mathbf{1}\left\{S_n + \frac{3}{2\sqrt{2\log 2}}\log n \in [x-1,x]\right\}\\ &\mathbf{1}\left\{S_l + S'_{n-l} + \frac{3}{2\sqrt{2\log 2}}\log n \in [x-1,x]\right\}\\ &\mathbf{1}\left\{S_k \leq \frac{k}{n}\left(x - \frac{3}{2\sqrt{2\log 2}}\log n\right) \text{ for } k\leq n\right\}\\ &\mathbf{1}\left\{S_l + S'_k \leq \frac{l+k}{n}\left(x - \frac{3}{2\sqrt{2\log 2}}\log n\right) \text{ for } k \leq n-l\right\}\Big).\end{aligned}$$

For $l = n$, we get simply $\mathbb{E}(W_n)$ so we will focus on the part of the sum for which $l < n$. Denote by s_l the summand. We will also condition on the value of S_l to make use of independence. Let $I_{l,j} = \frac{l}{n}(x - \frac{3}{2\sqrt{2\log 2}}\log n) + (-j-1,-j]$. We will condition on $S_l \in I_{l,j}$ and sum over j. We also note that $S_k - S_l$ is independent of S_l for $k > l$ so we find (using the conditions for the end points S_n and S'_{n-l})

$$\begin{aligned} s_l \leq &\sum_{j=0}^{\infty} C\left(n^{\frac{3}{2}}e^{-\sqrt{2\log 2}x}\right)^2\\ &\mathbb{E}\left(e^{\sqrt{2\log 2}S_l}\mathbf{1}\left\{S_l\in I_{l,j}, S_k \leq \frac{k}{n}\left(x - \frac{3}{2\sqrt{2\log 2}}\log n\right) \text{ for } k\leq l\right\}\right)\\ &\max_{z\in I_{l,j}} \mathbb{P}\Big(z + S_k \leq \frac{l+k}{n}\left(x - \frac{3}{2\sqrt{2\log 2}}\log n\right), \text{ for } k \leq n-l,\\ &\quad z + S_{n-l} + \frac{3}{2\sqrt{2\log 2}}\log n \in [x-1,x]\Big)^2\\ \leq &\sum_{j=0}^{\infty} Ce^{-\sqrt{2\log 2}j} n^{\frac{3}{2}(2-\frac{l}{n})} e^{-\sqrt{2\log 2}x(2-\frac{l}{n})}\\ &\mathbb{P}\left(S_l \in I_{l,j}; S_k \leq \frac{k}{n}\left(x - \frac{3}{2\sqrt{2\log 2}}\log n\right) \text{ for } k \leq l\right)\\ &\max_{z\in I_{l,j}} \mathbb{P}\Big(z + S_k \leq \frac{l+k}{n}\left(x - \frac{3}{2\sqrt{2\log 2}}\log n\right) \text{ for } k \leq n-l,\\ &\quad z + S_{n-l} + \frac{3}{2\sqrt{2\log 2}}\log n \in [x-1,x]\Big)^2.\end{aligned}$$

Let us estimate the two probabilities:

$$
\begin{aligned}
&\mathbb{P}\left(S_l \in I_{l,j}; S_k \leq \frac{k}{n}\left(x - \frac{3}{2\sqrt{2\log 2}}\log n\right) \text{ for } k \leq l\right)\\
&= \int_{I_{l,j}} dy \frac{e^{-\frac{y^2}{2l}}}{\sqrt{2\pi l}} \mathbb{P}_l^{0,z}\left(Y_k \leq \frac{k}{l}\left(\frac{l}{n}x - \frac{l}{n}\frac{3}{2\sqrt{2\log 2}}\log n\right) \text{ for } k \leq l\right)\\
&= \int_{I_{l,j}} dy \frac{e^{-\frac{y^2}{2l}}}{\sqrt{2\pi l}} \mathbb{P}_l^{0,z-\frac{l}{n}x+\frac{l}{n}\frac{3}{2\sqrt{2\log 2}}\log n}(Y_k \leq 0 \text{ for } k \leq l)\\
&\leq \int_{I_{l,j}} dy \frac{e^{-\frac{y^2}{2l}}}{\sqrt{2\pi l}} \mathbb{P}_l^{0,-j-1}(Y_k \leq 0 \text{ for } k \leq l)\\
&\leq C\frac{j+1}{l^{\frac{3}{2}}}.
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{P}\bigg(z + S_k \leq \frac{l+k}{n}\left(x - \frac{3}{2\sqrt{2\log 2}}\log n\right), \text{ for } k \leq n-l,\\
&\qquad z + S_{n-l} + \frac{3}{2\sqrt{2\log 2}}\log n \in [x-1, x]\bigg)\\
&= \int_{x-\frac{3}{2\sqrt{2\log 2}}\log n - z - 1}^{x-\frac{3}{2\sqrt{2\log 2}}\log n - z} dy \frac{e^{-\frac{y^2}{2(n-l)}}}{\sqrt{2\pi(n-l)}}\\
&\qquad \times \mathbb{P}_{n-l}^{0,y}\left(Y_k \leq \frac{l+k}{n}\left(x - \frac{3}{2\sqrt{2\log 2}}\log n\right) - z \text{ for } k \leq n-l\right)\\
&= \int_{x-\frac{3}{2\sqrt{2\log 2}}\log n - z - 1}^{x-\frac{3}{2\sqrt{2\log 2}}\log n - z} dy \frac{e^{-\frac{y^2}{2(n-l)}}}{\sqrt{2\pi(n-l)}}\\
&\mathbb{P}_{n-l}^{z-\frac{l}{n}(x-\frac{3}{2\sqrt{2\log 2}}\log n), y-x+\frac{3}{2\sqrt{2\log 2}}\log n + z - \frac{l}{n}(x-\frac{3}{2\sqrt{2\log 2}}\log n)}(Y_k \leq 0 \text{ for } k \leq n-l).
\end{aligned}
$$

We note that the starting point of the Brownian bridge can be estimated downwards to $-j-1$. Since $x \in [1, \log n]$ and $y - x + \frac{3}{2\sqrt{2\log 2}}\log n + z \in [-1, 0]$, we see that the endpoint can be estimated downwards to -1. Thus we can bound the probability from above by $C(j+1)(n-l)^{-\frac{3}{2}}$.

Collecting our estimates,

$$
\begin{aligned}
s_l &\leq \sum_{j=0}^{\infty} C e^{-\sqrt{2\log 2}j} n^{\frac{3}{2}(2-\frac{l}{n})} e^{-\sqrt{2\log 2}x(2-\frac{l}{n})}(j+1)^3 l^{-\frac{3}{2}}(n-l)^{-3}\\
&\leq C n^{\frac{3}{2}(2-\frac{l}{n})} e^{-\sqrt{2\log 2}x(2-\frac{l}{n})} l^{-\frac{3}{2}}(n-l)^{-3}.
\end{aligned}
$$

Switching $l \to n-l$ in the sum over l, we find

$$\begin{aligned}\mathbb{E}(Y_n^2) &\le \mathbb{E}(Y_n) + C\sum_{l=1}^{n-1} n^{\frac{3}{2}(1+\frac{l}{n})}e^{-\sqrt{2\log 2}x(1+\frac{l}{n})}(n-l)^{-\frac{3}{2}}l^{-3}\\ &\le \mathbb{E}(Y_n) + Ce^{-\sqrt{2\log 2}x}\sum_{l=1}^{n-1} n^{\frac{3}{2}(1+\frac{l}{n})}e^{-\sqrt{2\log 2}x\frac{l}{n}}(n-l)^{-\frac{3}{2}}l^{-3}\\ &\le \mathbb{E}(Y_n) + Ce^{-\sqrt{2\log 2}x}\sum_{l=1}^{n-1} n^{\frac{3}{2}(1+\frac{l}{n})}(n-l)^{-\frac{3}{2}}l^{-3}.\end{aligned}$$

Let us analyze the sum in detail. Write $f(x) = n^{\frac{3}{2}x}x^{-3}(1-x)^{-\frac{3}{2}}$. Then we can write the last sum as

$$n^{-3}\sum_{l=1}^{n-1} f\left(\frac{l}{n}\right) \tag{6.7}$$

which suggests estimating it as an integral. We note that f is decreasing in a regime $(0, \frac{1}{2} - o(1))$ and increasing in a regime $(\frac{1}{2} + o(1), 1)$ so we find

$$n^{-3}\sum_{l=1}^{n-1} f\left(\frac{l}{n}\right) \le n^{-3}f(n^{-1}) + 2n^{-2}\int_{\frac{1}{n}}^{1-\frac{1}{n}} f(x)dx + n^{-3}f(1-n^{-1}). \tag{6.8}$$

It is simple to check that the first and last terms are bounded as $n \to \infty$. For the integral, we note that for any fixed $\epsilon > 0$,

$$n^{-2}\int_{\epsilon}^{1-\epsilon} f(x)dx \to 0, \tag{6.9}$$

as $n \to \infty$, moreover for a fixed $\epsilon \in (0, \frac{1}{2})$

$$n^{-2}\int_{\frac{1}{\log n}}^{\epsilon} f(x)dx \le Cn^{-2+\frac{3}{2}\epsilon}(\log n)^3 \to 0 \tag{6.10}$$

as $n \to \infty$. We then note that

$$n^{-2}\int_{\frac{1}{n}}^{\frac{1}{\log n}} n^{\frac{3}{2}x}x^{-3}dx \le Cn^{-2}\int_{\frac{1}{n}}^{\frac{1}{\log n}} x^{-3}dx \le C \tag{6.11}$$

and

$$n^{-2}\int_{1-\epsilon}^{1-\frac{1}{n}} n^{\frac{3}{2}x}(1-x)^{-\frac{3}{2}}dx \le n^{-\frac{1}{2}}\int_{\frac{1}{n}}^{\epsilon} x^{-\frac{3}{2}}dx \le C \tag{6.12}$$

We conclude that $\mathbb{E}(Y_n^2) \le \mathbb{E}(Y_n) + Ce^{-\sqrt{2\log 2}x} \le C\mathbb{E}(Y_n)$ and

(6.13)
$$\mathbb{P}\left(\max_{\sigma\in\Sigma_n} X_\sigma \ge \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x\right) \ge \frac{(\mathbb{E}(Y_n))^2}{\mathbb{E}((Y_n)^2)} \ge Ce^{-\sqrt{2\log 2}x}.$$

2. Proof of the upper bound of the tail of the maximum

Our argument follows [**2**]. It is essentially a first moment estimate, but quite technical in that we keep careful track of the behavior of the random walk along the branch and not only its value at the tip. We will be slightly sloppy here - not differentiating between $\frac{n}{2}$ and its integer part. This would not make the argument conceptually any more difficult - only notationally unpleasant. We will begin with the following result:

LEMMA 6.4. *There exist constants $C_1 > 0$ and $C_2 > 0$ so that for any $n > 0$, $j \geq 0$, $x \geq 0$,*

$$\begin{aligned}
&\mathbb{P}\Bigg(\exists \sigma \in \Sigma_n : \ X_{\sigma|k} - \sqrt{2\log 2}k \leq 0 \text{ for } k \leq n, \\
&\qquad X_\sigma - \sqrt{2\log 2}n + \frac{3}{2\sqrt{2\log 2}} \log n - x \in [0,1) \\
&\qquad \max_{\frac{1}{2}n \leq k \leq n} (X_{\sigma|k} - \sqrt{2\log 2}k) + \frac{3}{2\sqrt{2\log 2}} \log n - x - j \in [0,1) \Bigg) \\
&\leq C_1 e^{-C_2 j} e^{-\sqrt{2\log 2}x}.
\end{aligned}$$

PROOF. Let us write E for the event in the lemma. Also write

$$d_k = d_k(n, x+j) = \mathbf{1}\left\{ \frac{1}{2}n \leq k \leq n \right\} \left(x + j + 1 - \frac{3}{2\sqrt{2\log 2}} \log n \right) \wedge 0. \tag{6.14}$$

We then note that since

$$\begin{aligned}
&\mathbf{1}\left\{ X_{\sigma|k} - \sqrt{2\log 2}k \leq 0, \text{for } k \leq n \right\} \\
&\times \mathbf{1}\left\{ \max_{\frac{1}{2}n \leq k \leq n} (X_{\sigma|k} - \sqrt{2\log 2}k) + \frac{3}{2\sqrt{2\log 2}} \log n - x - j \in [0,1) \right\} \\
&\leq \mathbf{1}\{ X_{\sigma|k} - \sqrt{2\log 2}k \leq d_k, \ \text{for } k \leq n \},
\end{aligned}$$

partitioning according to the maximum being obtained at time k gives, $E \subset \cup_{\frac{1}{2}n \leq k \leq n} E_k = \cup_{\frac{1}{2}n \leq k \leq n} \cup_{\sigma \in \Sigma_n} E_k(\sigma)$, where

$$\begin{aligned}
E_k(\sigma) = \Bigg\{ & X_{\sigma|l} - \sqrt{2\log 2}l \leq d_l, \ \text{for } l \leq n, \\
& X_\sigma - \sqrt{2\log 2}n + \frac{3}{2\sqrt{2\log 2}} \log n - x \in [0,1), \\
& X_{\sigma|k} - \sqrt{2\log 2}k + \frac{3}{2\sqrt{2\log 2}} \log n - x - j \in [0,1) \Bigg\}
\end{aligned}$$

and $E_k = \cup_{\sigma \in \Sigma_n} E_k(\sigma)$.

Anticipating the use of Theorem 2.1, we define

$$\begin{aligned}
E_k(S) = \Bigg\{ & S_l \leq d_l, \ \text{for } l \leq n, \ S_n + \frac{3}{2\sqrt{2\log 2}} \log n - x \in [0,1), \\
& S_k + \frac{3}{2\sqrt{2\log 2}} \log n - x - j \in [0,1) \Bigg\}.
\end{aligned}$$

So by Theorem 2.1,

$$\begin{aligned}\mathbb{P}(\cup_{\sigma\in\Sigma_n} E_k(\sigma)) &\leq \mathbb{E}(e^{-\sqrt{2\log 2}S_n}; E_k(S))\\ &\leq Cn^{\frac{3}{2}}e^{-\sqrt{2\log 2}x}\mathbb{P}(E_k(S)).\end{aligned}$$

By the Markov property of the random walk, and noting that on the event we are considering, $d_l - S_k \leq 1$ for $l \geq k$, we see that

$$\begin{aligned}\mathbb{P}(E_k(S)) \leq \mathbb{P}&\left(S_l \leq d_l,\ \text{for } l \leq k,\ S_k + \frac{3}{2\sqrt{2\log 2}}\log n - x - j \in [0,1)\right)\\ &\times \mathbb{P}\left(S_{n-k} \in [-j-1, -j+1],\ \max_{l\leq n-k} S_l \leq 1\right).\end{aligned}$$

For the second term, we see by our standard gambler's ruin estimates that

$$\begin{aligned}&\mathbb{P}\left(S_{n-k} \in [-j-1, -j+1],\ \max_{l\leq n-k} S_l \leq 1\right)\\ &= \int_{-j-1}^{-j+1} dy \frac{e^{-\frac{y^2}{2(n-k)}}}{\sqrt{2\pi(n-k)}}\mathbb{P}^{0,y}_{n-k}(Y_l \leq 1 \text{ for } l \leq n-k)\\ &\leq C(j+1)(n-k+1)^{-\frac{3}{2}}.\end{aligned}$$

The first term we consider in two different cases. First let $\frac{3}{4}n \leq k \leq n$. We then have

$$\begin{aligned}&\mathbb{P}\left(S_l \leq d_l,\ \text{for } l \leq k,\ S_k + \frac{3}{2\sqrt{2\log 2}}\log n - x - j \in [0,1)\right)\\ &\leq \int_{-\infty}^{x+j+1-\frac{3}{2\sqrt{2\log 2}}\log n} dy \frac{e^{-\frac{y^2}{n}}}{\sqrt{\pi n}}\mathbb{P}^{0,y}_{\frac{1}{2}n}\left(Y_l \leq 0,\ \text{for } l \leq \frac{1}{2}n\right)\int_{x+j-\frac{3}{2\sqrt{2\log 2}}\log n - y}^{x+j-\frac{3}{2\sqrt{2\log 2}}\log n-y+1} dz\\ &\quad\times \frac{e^{-\frac{z^2}{2(k-\frac{1}{2}n)}}}{\sqrt{2\pi(k-\frac{1}{2}n)}}\mathbb{P}^{y,y+z}_{k-\frac{1}{2}n}\left(Y_l \leq x+j+1-\frac{3}{2\sqrt{2\log 2}}\log n,\ \text{for } l \leq k-\frac{1}{2}n\right)\\ &\leq \int_{-\infty}^{x+j+1-\frac{3}{2\sqrt{2\log 2}}\log n} dy \frac{e^{-\frac{y^2}{n}}}{\sqrt{\pi n}}\mathbb{P}^{0,y-1}_{\frac{1}{2}n}\left(Y_l \leq 0 \text{ for } l \leq \frac{1}{2}n\right)\\ &\quad\times \int_{x+j-\frac{3}{2\sqrt{2\log 2}}\log n-y}^{x+j-\frac{3}{2\sqrt{2\log 2}}\log n-y+1} dz \frac{e^{-\frac{z^2}{2k-n}}}{\sqrt{\pi(2k-n)}}\\ &\quad\times \mathbb{P}^{y-(x+j+1-\frac{3}{2\sqrt{2\log 2}}\log n),y+z-(x+j+1-\frac{3}{2\sqrt{2\log 2}}\log n)}_{k-\frac{1}{2}n}\left(Y_l \leq 0, \text{for } l \leq k-\frac{1}{2}n\right).\end{aligned}$$

We note that first of all, we can restrict to the case that $x+j < \frac{3}{2\sqrt{2\log 2}}\log n$, since otherwise the original probability is zero. Thus $y-1$ takes on only negative values and the first Brownian bridge probability can be estimated upwards to $C(2-y)n^{-1}$. We then note that we can estimate the end point of the second Brownian bridge downwards to -1 (this is simply due to the integration region of z). The starting point can be estimated downwards to $y-1$ (for the reason that $x+j < \frac{3}{2\sqrt{2\log 2}}\log n$). Thus the second Brownian bridge probability is less than $C(2-y)(k-\frac{1}{2}n)^{-1}$.

Estimating the z-exponential upwards to one, noting that the z-interval is of unit length and once again using $x+j-\frac{3}{2\sqrt{2\log 2}}\log n<0$,

$$\begin{aligned}
&\mathbb{P}\left(S_l \le d_l,\ \text{for } l\le k,\ S_k+\frac{3}{2\sqrt{2\log 2}}\log n - x - j\in[0,1)\right)\\
&\le C\left(k-\frac{1}{2}n\right)^{-\frac{3}{2}}\int_{-\infty}^{1} dy\frac{e^{-\frac{y^2}{n}}}{\sqrt{\pi n}}C(2-y)^2n^{-1}\\
&\le C\left(k-\frac{1}{2}n\right)^{-\frac{3}{2}}\\
&\le Cn^{-\frac{3}{2}}.
\end{aligned}$$

On the other hand, for $\frac{1}{2}n\le k\le\frac{3}{4}n$, using naive estimates ($d_l\le 0$, $x\ge 0$ and $j\ge 0$)

$$\begin{aligned}
&\mathbb{P}\left(S_l \le d_l,\ \text{for } l\le k,\ S_k+\frac{3}{2\sqrt{2\log 2}}\log n - x - j\in[0,1)\right)\\
&\le \mathbb{P}\left(S_l \le 0,\ \text{for } l\le k,\ S_k+\frac{3}{2\sqrt{2\log 2}}\log n - x - j\in[0,1)\right)\\
&=\int_{x+j-\frac{3}{2\sqrt{2\log 2}}\log n}^{x+j-\frac{3}{2\sqrt{2\log 2}}\log n+1}\frac{e^{-\frac{y^2}{2k}}}{\sqrt{2\pi k}}dy\mathbb{P}_k^{0,y}(Y_l\le 0,\ \text{for } l\le k)\\
&\le\int_{x+j-\frac{3}{2\sqrt{2\log 2}}\log n}^{x+j-\frac{3}{2\sqrt{2\log 2}}\log n+1}\frac{e^{-\frac{y^2}{2k}}}{\sqrt{2\pi k}}dy\mathbb{P}_k^{0,-C\log n}(Y_l\le 0 \text{ for } l\le k)\\
&\le C\log n k^{-\frac{3}{2}}\\
&\le C\log n n^{-\frac{3}{2}}\\
&\le C(1+j)\log n n^{-\frac{3}{2}}.
\end{aligned}$$

We are still missing smallness in j. We introduce a parameter $a\in[1,\frac{n}{4}]$ which we shall tune in our estimates so that we get the smallness in j. So far we have estimated $\mathbb{P}(E_k(S))$ and we have

$$\begin{aligned}
&n^{\frac{3}{2}}(j+1)^{-2}\sum_{\frac{1}{2}n\le k\le n-a}\mathbb{P}(E_k(S))\\
&\le C\sum_{\frac{1}{2}n\le k\le n-a}(n-k+1)^{-\frac{3}{2}}\left(\mathbf{1}\left\{k\le\frac{3}{4}n\right\}\log n+\mathbf{1}\left\{k\ge\frac{3}{4}n\right\}\right)\\
&\le C\sum_{\frac{1}{2}n\le k\le n-a}\left(\mathbf{1}\left\{k\le\frac{3}{4}n\right\}n^{-\frac{3}{2}}\log n+\frac{1}{(n-k+1)^{\frac{3}{2}}}\mathbf{1}\left\{k\ge\frac{3}{4}n\right\}\right).
\end{aligned}$$

The first term we can estimate upwards to something proportional to $\frac{\log n}{\sqrt{n}}$. The sum in the second term we estimate upwards to $\sum_{k=a}^{\infty}\frac{1}{(1+k)^{\frac{3}{2}}}$ which is of order $a^{-\frac{1}{2}}$. Thus

$$\sum_{\frac{1}{2}n\leq k\leq n-a} \mathbb{P}(E_k) \leq C(j+1)^2(a^{-\frac{1}{2}}+n^{-\frac{1}{2}}\log n)e^{-\sqrt{2\log 2}x}. \tag{6.15}$$

We still need to analyze $\mathbb{P}(E_k)$ for $k \geq n-a$. To do this, we note that by Theorem 2.1 we have for $k \geq \frac{3}{4}n$.

$$\begin{aligned}\mathbb{P}(E_k) \leq \mathbb{P}\Bigg(&\exists\sigma\in\Sigma_k:\ (X_{\sigma|l}-\sqrt{2\log 2}l)\leq d_l,\ \text{for } l\leq k,\\ &X_\sigma-\sqrt{2\log 2}k+\frac{3}{2\sqrt{2\log 2}}\log n - x - j\in[0,1)\Bigg)\\ \leq Cn^{\frac{3}{2}}e^{-\sqrt{2\log 2}x}e^{-\sqrt{2\log 2}j}\mathbb{P}&\left(S_k\leq d_l,\ \text{for } l\leq k, S_k+\frac{3}{2\sqrt{2\log 2}}\log n-x-j\in[0,1)\right).\end{aligned}$$

We already estimated this last probability and found an upper bound of the form $Cn^{-\frac{3}{2}}(1+j)$ so we see that for $k \geq \frac{3}{4}n$

$$\mathbb{P}(E_k) \leq C(1+j)e^{-\sqrt{2\log 2}j}e^{-\sqrt{2\log 2}x}. \tag{6.16}$$

We conclude that

$$\sum_{n-a\leq k\leq n} \mathbb{P}(E_k) \leq C(1+a)(1+j)e^{-\sqrt{2\log 2}x}e^{-\sqrt{2\log 2}j} \tag{6.17}$$

and

(6.18)
$$\mathbb{P}(E) \leq Ce^{-\sqrt{2\log 2}x}\left((j+1)^2\left(a^{-\frac{1}{2}}+\frac{\log n}{\sqrt{n}}\right)+e^{-\sqrt{2\log 2}j}(1+j)(a+1)\right).$$

Now by optimizing over a, we will be able to recover the claim. First of all, we note that if $j > \frac{3}{2\sqrt{2\log 2}}\log n$, then the probability we are considering is zero so we may consider only $j \leq \frac{3}{2\sqrt{2\log 2}}\log n$. In this case, $\frac{\log n}{\sqrt{n}} \leq e^{-\epsilon j}$ for some small enough positive ϵ. So if we take $a = \max(1, \alpha e^{\beta j})$ for some small α and β, we find our claim.

□

As a corollary we get a result quite similar to the one we are looking for. It says that if we consider a branching random walk which we kill when it hits the barrier $\sqrt{2\log 2}|\sigma|$, then we have an exponential upper bound for the corresponding term. Since we know that $\max_{\sigma\in\Sigma_n} X_\sigma - \sqrt{2\log 2}n \to -\infty$ almost surely, this killing shouldn't affect the process that much. After this corollary, the rest of the proof is to make this statement rigorous.

Corollary 6.5. *There is a constant $C > 0$ such that for all $x \in \mathbb{R}$,*

$$\mathbb{P}\left(\exists\sigma\in\Sigma_n:\ X_\sigma\geq\sqrt{2\log 2}n-\frac{3}{2\sqrt{2\log 2}}\log n+x, X_{\sigma|k}-\sqrt{2\log 2}k\leq 0,\ \text{for } k\leq n\right)$$
$$\leq Ce^{-\sqrt{2\log 2}x}.$$

PROOF. For $x \leq 0$ the result is trivial. For $x \geq 0$ summing over j in Lemma 6.4, we get

$$\mathbb{P}\left(\exists \sigma \in \Sigma_n :\ X_\sigma - \sqrt{2\log 2}n + \frac{3}{2\sqrt{2\log 2}}\log n \in [x, x+1], X_{\sigma|k} - \sqrt{2\log 2}k \leq 0,\ \text{for } k \leq n\right)$$
$$\leq Ce^{-\sqrt{2\log 2}x}$$

Substituting $x \to x + i$ and summing over i, we find the claim.

□

Let us introduce some notation:

$$\mathcal{S}_k^r = \left\{\sigma \in \Sigma_k :\ \max_{l \leq k-1}(X_{\sigma|l} - \sqrt{2\log 2}l) \leq X_\sigma - \sqrt{2\log 2}k \leq r\right\}, \tag{6.19}$$

So this is the (random) set of all branches σ of length k such that $X_{\sigma|l} - \sqrt{2\log 2}l$ attains its maximum at $l = k$ and this maximum is less than r. Let $\mathcal{S}^r = \cup_k \mathcal{S}_k^r$.

For $\sigma \in \cup_{k \leq n}\Sigma_n$ let us write $B_n^x(\sigma) = 1$ if there is a $\sigma' \in \Sigma_n$ such that $\sigma \leq \sigma'$, $\max_{|\sigma| \leq l \leq n}(X_{\sigma'|l} - \sqrt{2\log 2}l) \leq X_\sigma - \sqrt{2\log 2}|\sigma|$ and $X_{\sigma'} - \sqrt{2\log 2}n \geq -\frac{3}{2\sqrt{2\log 2}}\log n + x$. Otherwise, let $B_n^x(\sigma) = 0$. So in words, $B_n^x(\sigma)$ indicates if there is a branch $X_{\sigma'}$ such that $X_{\sigma'}$ lives on the scale we expect the maximum to live on, if σ' is a descendant of σ and if after σ, $X_{\sigma'|l} - \sqrt{2\log 2}l$ does not go above $X_\sigma - \sqrt{2\log 2}|\sigma|$.

We shall need the following lemma

LEMMA 6.6. *There exists a constant $C > 0$ such that for $x \in [1, \frac{3}{2\sqrt{2\log 2}}\log n]$*

$$\mathbb{E}\left(\sum_{\sigma \in \mathcal{S}^x} B_n^x(\sigma)\right) \leq Cxe^{-\sqrt{2\log 2}x}. \tag{6.20}$$

PROOF. Let $k \leq \frac{n}{2}$ and $\widetilde{X}$ be an independent copy of X (quantities with $\widetilde{\cdot}$ will refer to $\widetilde{X}$ being used in the definition: $\widetilde{\mathbb{E}}$ is averaging only over $\widetilde{X}$, $\widetilde{\mathcal{S}}$ is defined using $\widetilde{X}$ and so on) and consider

$$\widetilde{\mathbb{E}}\Bigg(\sum_{\sigma \in \widetilde{\mathcal{S}}_k^x} \mathbb{P}\Bigg(\exists \sigma' \in \Sigma_{n-k} :\ X_{\sigma'} \geq \sqrt{2\log 2}(n-k) - \frac{3}{2\sqrt{2\log 2}}\log(n-k)$$
$$+ x - 1 + \sqrt{2\log 2}k - \widetilde{X}_\sigma,\ X_{\sigma'|l} - \sqrt{2\log 2}l \leq 0,\ \text{for } l \leq n-k\Bigg)\Bigg)$$
$$= \widetilde{\mathbb{E}}\Bigg(\mathbb{E}\Bigg(\sum_{\sigma \in \widetilde{\mathcal{S}}_k^x} \mathbf{1}\Bigg\{\exists \sigma' \in \Sigma_{n-k} :\ X_{\sigma'} + \widetilde{X}_\sigma \geq \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n$$
$$+ \frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k} + x - 1,\ X_{\sigma'|l} - \sqrt{2\log 2}l \leq 0,\ \text{for } l \leq n-k\Bigg\}\Bigg)\Bigg).$$

For large enough n (making use of $k \leq \frac{n}{2}$), we see that $0 \leq \frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k} \leq 1$ so

$$\widetilde{\mathbb{E}}\Bigg(\mathbb{E}\Bigg(\sum_{\sigma\in\widetilde{\mathcal{S}}_k^x}\mathbf{1}\Big\{\exists\sigma'\in\Sigma_{n-k}:\ X_{\sigma'}+\widetilde{X}_\sigma\geq\sqrt{2\log 2}n-\frac{3}{2\sqrt{2\log 2}}\log n$$

$$+\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k}+x-1,\ X_{\sigma'|l}-\sqrt{2\log 2}l\leq 0,\ \text{for } l\leq n-k\Big\}\Bigg)\Bigg)$$

$$\geq\widetilde{\mathbb{E}}\Bigg(\mathbb{E}\Bigg(\sum_{\sigma\in\widetilde{\mathcal{S}}_k^x}\mathbf{1}\Big\{\exists\sigma'\in\Sigma_{n-k}:\ X_{\sigma'}+\widetilde{X}_\sigma\geq\sqrt{2\log 2}n-\frac{3}{2\sqrt{2\log 2}}\log n$$

$$+x,\ X_{\sigma'|l}-\sqrt{2\log 2}l\leq 0,\ \text{for } l\leq n-k\Big\}\Bigg)\Bigg)$$

$$=\widetilde{\mathbb{E}}\left(\sum_{\sigma\in\widetilde{\mathcal{S}}_k^x}\widetilde{B}_n^x(\sigma)\right).$$

Making use of Corollary 6.5, we see that

$$\mathbb{P}\Big(\exists\sigma'\in\Sigma_{n-k}:\ X_{\sigma'}\geq\sqrt{2\log 2}(n-k)-\frac{3}{2\sqrt{2\log 2}}\log(n-k)$$

$$+x-1+\sqrt{2\log 2}k-\widetilde{X}_\sigma,\ X_{\sigma'|l}-\sqrt{2\log 2}l\leq 0,\ \text{for } l\leq n-k\Big)$$

$$\leq Ce^{-\sqrt{2\log 2}(x+\sqrt{2\log 2}k-\widetilde{X}_\sigma)}.$$

We conclude that

$$\mathbb{E}\left(\sum_{\sigma\in\mathcal{S}_k^x}B_n^x(\sigma)\right)\leq e^{-\sqrt{2\log 2}x}\mathbb{E}\left(\sum_{\sigma\in\mathcal{S}_k^x}e^{\sqrt{2\log 2}(X_\sigma-\sqrt{2\log 2}k)}\right). \tag{6.21}$$

From the definition of $\mathcal{S}_k^x$ and Theorem 2.1, we see that

$$\mathbb{E}\left(\sum_{\sigma\in\mathcal{S}_k^x}e^{\sqrt{2\log 2}(X_\sigma-\sqrt{2\log 2}k)}\right)=\mathbb{P}(S_l\leq S_k\leq x,\ \text{for } l\leq k). \tag{6.22}$$

We note that summing this last probability over all k (and using $(S_k-S_l)\overset{d}{=}(S_{k-l})$) just gives the renewal function $R(x)$ so summing over $k\leq\frac{n}{2}$ gives

$$\mathbb{E}\left(\sum_{\sigma\in\mathcal{S}^x}B_n^x(\sigma)\mathbf{1}\left\{|\sigma|\leq\frac{n}{2}\right\}\right)\leq R(x)e^{-\sqrt{2\log 2}x}. \tag{6.23}$$

For $n>|\sigma|>\frac{n}{2}$, we split $B_n^x(\sigma)$ into two parts: those for which $X_\sigma-\sqrt{2\log 2}|\sigma|\leq x-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-|\sigma|}$ and those for which the opposite inequality holds. For $n>k>\frac{n}{2}$ we have (using similar notation and reasoning as before)

$$
\begin{aligned}
&\mathbb{E}\left(\sum_{\sigma\in\mathcal{S}_k^x} B_n^x(\sigma)\mathbf{1}\left\{X_\sigma-\sqrt{2\log 2}k\le x-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k}\right\}\right)\\
&=\widetilde{\mathbb{E}}\Bigg(\sum_{\sigma\in\widetilde{\mathcal{S}}_k^x}\mathbb{E}\Bigg(\mathbf{1}\Bigg\{\exists\sigma'\in\Sigma_{n-k}:\ \widetilde{X}_\sigma+X_{\sigma'}\ge\sqrt{2\log 2}n-\frac{3}{2\sqrt{2\log 2}}\log n+x,\\
&\qquad X_{\sigma'|l}-\sqrt{2\log 2}l\le 0,\ \text{for } l\le n-k\Bigg\}\Bigg)\\
&\qquad\times\mathbf{1}\left\{\widetilde{X}_\sigma-\sqrt{2\log 2}k\le x-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k}\right\}\Bigg).
\end{aligned}
$$

We note that Corollary 6.5 implies that

$$
\begin{aligned}
&\mathbb{E}\Bigg(\mathbf{1}\Bigg\{\exists\sigma'\in\Sigma_{n-k}:\ \widetilde{X}_\sigma+X_{\sigma'}\ge\sqrt{2\log 2}n-\frac{3}{2\sqrt{2\log 2}}\log n+x,\\
&\qquad X_{\sigma'|l}-\sqrt{2\log 2}l\le 0,\ \text{for } l\le n-k\Bigg\}\Bigg)\\
&=\mathbb{E}\Bigg(\mathbf{1}\Bigg\{\exists\sigma'\in\Sigma_{n-k}:\ X_{\sigma'}\ge\sqrt{2\log 2}(n-k)-\frac{3}{2\sqrt{2\log 2}}\log(n-k)+x\\
&-\widetilde{X}_\sigma+\sqrt{2\log 2}k-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k},X_{\sigma'|l}-\sqrt{2\log 2}l\le 0,\ \text{for } l\le n-k\Bigg\}\Bigg)\\
&\le C\left(\frac{n}{n-k}\right)^{\frac{3}{2}}e^{-\sqrt{2\log 2}(x-\widetilde{X}_\sigma+\sqrt{2\log 2}k)}.
\end{aligned}
$$

We thus find by Theorem 2.1

$$
\begin{aligned}
&\mathbb{E}\left(\sum_{\sigma\in\mathcal{S}_k^x} B_n^x(\sigma)\mathbf{1}\left\{X_\sigma-\sqrt{2\log 2}k\le x-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k}\right\}\right)\\
&\le Ce^{-\sqrt{2\log 2}x}\left(\frac{n}{n-k}\right)^{\frac{3}{2}}\mathbb{E}\Bigg(\sum_{\sigma\in\mathcal{S}_k^x}e^{\sqrt{2\log 2}(\widetilde{X}_\sigma-\sqrt{2\log 2}k)})\\
&\times\mathbf{1}\left\{X_\sigma-\sqrt{2\log 2}k\le x-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k}\right\}\Bigg)\\
&=Ce^{-\sqrt{2\log 2}x}\left(\frac{n}{n-k}\right)^{\frac{3}{2}}\mathbb{P}\left(S_l\le S_k\le x-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k},\ \text{for } l\le k\right).
\end{aligned}
$$

For the probability, we find

$$\begin{aligned}
&\int_{-\infty}^{x-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k}} dy \frac{e^{-\frac{y^2}{2k}}}{\sqrt{2\pi k}} \mathbb{P}_k^{0,y}(Y_l \le y \text{ for } l \le k) \\
&= \int_{-\infty}^{x-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k}} dy \frac{e^{-\frac{y^2}{2k}}}{\sqrt{2\pi k}} \mathbb{P}_k^{-y,0}(Y_l \le 0 \text{ for } l \le k) \\
&= \int_{0}^{0\vee(x-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k})} dy \frac{e^{-\frac{y^2}{2k}}}{\sqrt{2\pi k}} \mathbb{P}_k^{-y,0}(Y_l \le 0 \text{ for } l \le k) \\
&\le Ck^{-\frac{3}{2}} \int_{0}^{0\vee(x-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k})} dy(1+y) \\
&\le Ck^{-\frac{3}{2}} \left(1 + 0 \vee \left(x - \frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k}\right)\right)^2.
\end{aligned}$$

Since $x \le \frac{3}{2\sqrt{2\log 2}}\log n$, we see that the probability is less than

$$Ck^{-\frac{3}{2}}\left(1+\log(n-k)\right)^2 \tag{6.24}$$

and

$$\begin{aligned}
&\mathbb{E}\left(\sum_{\sigma\in\mathcal{S}_k^x} B_n^x(\sigma)\mathbf{1}\left\{X_\sigma - \sqrt{2\log 2}k \le x - \frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k}\right\}\right) \\
&\le Ce^{-\sqrt{2\log 2}x}\left(\frac{n}{n-k}\right)^{\frac{3}{2}} k^{-\frac{3}{2}}(1+\log(n-k)^2).
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{E}\left(\sum_{\sigma\in\mathcal{S}^x} B_n^x(\sigma)\mathbf{1}\left\{n > |\sigma| > \frac{n}{2}\right\}\mathbf{1}\left\{X_\sigma - \sqrt{2\log 2}|\sigma| \le x - \frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-|\sigma|}\right\}\right) \\
&\le \sum_{k=\frac{n}{2}}^{n-1} Ce^{-\sqrt{2\log 2}x}\left(\frac{n}{n-k}\right)^{\frac{3}{2}} k^{-\frac{3}{2}}(1+\log(n-k)^2) \\
&\le Ce^{-\sqrt{2\log 2}x}\sum_{k=1}^{\infty}\frac{(1+\log k)^2}{k^{\frac{3}{2}}} \\
&\le Ce^{-\sqrt{2\log 2}x}.
\end{aligned}$$

Next we estimate (by a trivial estimate $B_n^x(\sigma) \le 1$ and Theorem 2.1) for $\frac{n}{2} < k < n$

$$
\begin{aligned}
&\mathbb{E}\left(\sum_{\sigma\in\mathcal{S}_k^x} B_n^x(\sigma)\mathbf{1}\left\{X_\sigma-\sqrt{2\log 2}k\geq x-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k}\right\}\right)\\
&\leq \mathbb{E}\left(\sum_{\sigma\in\mathcal{S}_k^x} \mathbf{1}\left\{X_\sigma-\sqrt{2\log 2}k\geq x-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k}\right\}\right)\\
&\leq \mathbb{E}\left(e^{-\sqrt{2\log 2}S_k}; S_l\leq S_k\leq x \text{ for } l\leq k,\ S_k\geq x-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k}\right)\\
&=\int_{x-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k}}^{x} dy\frac{e^{-\frac{y^2}{2k}}}{\sqrt{2\pi k}}e^{-\sqrt{2\log 2}y}\mathbb{P}_k^{0,y}(Y_l\leq y,\ \text{for } l\leq k)\\
&\leq C\int_{0\vee(x-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k})}^{\infty} k^{-\frac{3}{2}}e^{-\sqrt{2\log 2}y}(1+y)dy\\
&\leq Ce^{-\sqrt{2\log 2}(0\vee(x-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k}))}k^{-\frac{3}{2}}\left(1+0\vee\left(x-\frac{3}{2\sqrt{2\log 2}}\log\frac{n}{n-k}\right)\right)\\
&\leq Ce^{-\sqrt{2\log 2}x}\left(\frac{n}{n-k}\right)^{\frac{3}{2}}k^{-\frac{3}{2}}(1+\log(n-k))\\
&\leq Ce^{-\sqrt{2\log 2}x}(n-k)^{-\frac{3}{2}}(1+\log(n-k)),
\end{aligned}
$$

where we again used $x\leq\frac{3}{2\sqrt{2\log 2}}\log n$. This again is summable over $\frac{n}{2}<k<n$. We conclude

$$
\mathbb{E}\left(\sum_{\sigma\in\mathcal{S}^x} B_n^x(\sigma)\mathbf{1}\left\{n>|\sigma|>\frac{n}{2}\right\}\right)\leq Ce^{-\sqrt{2\log 2}x}. \tag{6.25}
$$

Finally we note that by Theorem 2.1

$$
\begin{aligned}
&\mathbb{E}\left(\sum_{\sigma\in\mathcal{S}_n^x} B_n^x(\sigma)\right)\\
&= \sum_{\sigma\in\Sigma_n} \mathbb{E}\Bigg(\mathbf{1}\Big\{X_{\sigma|l} - \sqrt{2\log 2}l \le X_\sigma - \sqrt{2\log 2}n \le x,\ \text{for } l\le n\Big\}\\
&\qquad\qquad \mathbf{1}\Big\{\ X_\sigma \ge \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x\Big\}\Bigg)\\
&= \mathbb{E}\left(e^{-\sqrt{2\log 2}S_n}; S_l \le S_n \le x,\ \text{for } l\le n, S_n \ge x - \frac{3}{2\sqrt{2\log 2}}\log n\right)\\
&= \int_{x-\frac{3}{2\sqrt{2\log 2}}\log n}^{x} dy \frac{e^{-\frac{y^2}{2n}}}{\sqrt{2\pi n}} e^{-\sqrt{2\log 2}y}\mathbb{P}_n^{0,y}(Y_l \le y,\ \text{for } l\le n)\\
&= \int_0^x dy \frac{e^{-\frac{y^2}{2n}}}{\sqrt{2\pi n}} e^{-\sqrt{2\log 2}y}\mathbb{P}_n^{-y,0}(Y_l \le 0,\ \text{for } l\le n)\\
&\le Cn^{-\frac{3}{2}}\int_0^\infty (1+y)e^{-\sqrt{2\log 2}y}dy\\
&\le Cn^{-\frac{3}{2}}\\
&\le Ce^{-\sqrt{2\log 2}x}
\end{aligned}
$$

since for $x \le \frac{3}{2\sqrt{2\log 2}}\log n$, $n^{-\frac{3}{2}} \le Ce^{-\sqrt{2\log 2}x}$. As $\frac{R(x)}{x} \to c_0$ as $x\to\infty$, we conclude that for $x \in [1, \frac{3}{2\sqrt{2\log 2}}\log n]$

$$
\mathbb{E}\left(\sum_{\sigma\in\mathcal{S}^x} B_n^x(\sigma)\right) \le Cxe^{-\sqrt{2\log 2}x}. \tag{6.26}
$$

□

We are now in a position to prove Proposition 6.2:

Proof of Proposition 6.2. We begin by noting that on the event

$$
\{X_\sigma - \sqrt{2\log 2}|\sigma| \le x \text{ for all } \sigma \in \cup_k\Sigma_k\}, \tag{6.27}
$$

$$
\left\{\sum_{\sigma\in\mathcal{S}^x} B_n^x(\sigma) \ge 1\right\} = \left\{\exists\sigma'\in\Sigma_n:\ X_{\sigma'} \ge \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x\right\}. \tag{6.28}
$$

This is due to the fact that the condition $X_\sigma - \sqrt{2\log 2}|\sigma| \le x$ is fulfilled automatically in the definition of $\mathcal{S}^x$. Thus

$$\begin{aligned}
&\mathbb{P}\left(\max_{\sigma\in\Sigma_n} X_\sigma \geq \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x\right)\\
&\leq \mathbb{P}\left(\left\{\sum_{\sigma\in\mathcal{S}^x} B_n^x(\sigma) \geq 1\right\} \cap \{X_\sigma - \sqrt{2\log 2}|\sigma| \leq x \text{ for all } \sigma \in \cup_k\Sigma_k\}\right)\\
&+ \mathbb{P}\left(\exists \sigma \in \cup_k\Sigma_k :\ X_\sigma - \sqrt{2\log 2}|\sigma| > x\right)\\
&\leq \mathbb{P}\left(\sum_{\sigma\in\mathcal{S}^x} B_n^x(\sigma) \geq 1\right) + \mathbb{P}\left(\exists \sigma \in \cup_k\Sigma_k :\ X_\sigma - \sqrt{2\log 2}|\sigma| > x\right)\\
&\leq \mathbb{E}\left(\sum_{\sigma\in\mathcal{S}^x} B_n^x(\sigma)\right) + \mathbb{P}\left(\exists \sigma \in \cup_k\Sigma_k :\ X_\sigma - \sqrt{2\log 2}|\sigma| > x\right).
\end{aligned}$$

In the last step, we used Markov's inequality. The first term gives a bound of the form we wish. For the second term, we have by Theorem 2.1 the estimate

$$\begin{aligned}
&\mathbb{P}\left(\exists \sigma \in \cup_k\Sigma_k :\ X_\sigma - \sqrt{2\log 2}|\sigma| > x\right)\\
&\leq \sum_{n=1}^{\infty} \mathbb{E}\left(\sum_{\sigma\in\Sigma_n} \mathbf{1}\left\{X_\sigma - \sqrt{2\log 2}n \geq x,\ X_{\sigma|k} - \sqrt{2\log 2}k < x,\ \text{for } k < n\right\}\right)\\
&= \sum_{n=0}^{\infty} \mathbb{E}\left(e^{-\sqrt{2\log 2}S_n}\mathbf{1}\left\{S_n \geq x,\ S_k < x,\ \text{for } k < n\right\}\right)\\
&\leq \sum_{n=1}^{\infty} \int_x^\infty dy \frac{e^{-\frac{y^2}{2n}}}{\sqrt{2\pi n}} e^{-\sqrt{2\log 2}y} \mathbb{P}_n^{0,y}\left(Y_l < x,\ \text{for } l < n\right)\\
&\leq \sum_{n=1}^{\infty} Cxe^{-\sqrt{2\log 2}x} n^{-\frac{3}{2}}\\
&\leq Cxe^{-\sqrt{2\log 2}x}.
\end{aligned}$$

Combining our estimates gives the desired result.

□

3. Proof of the upper bound for the tail of $W_{n,\beta}$

The proof if this is very similar to the proof of Proposition 6.2. The starting point is the following remark: for

$$\mathcal{F}_n := \left\{\frac{1}{\beta}\log\left(n^{\frac{3\beta}{2\sqrt{2\log 2}}} e^{\frac{1}{2}(\beta-\sqrt{2\log 2})^2 n} W_{n,\beta}\right) \geq x\right\} \tag{6.29}$$

$\mathbb{P}(\mathcal{F}_n) \leq \mathbb{P}(A_n) + \mathbb{P}(B_n) + \mathbb{P}(C_n) + \mathbb{P}(D_n)$, where

$$
\begin{aligned}
A_n &= \left\{ \max_{\sigma\in\Sigma_n} X_\sigma \geq \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x \right\} \\
B_n &= \left\{ (X_\sigma - \sqrt{2\log 2}|\sigma|) \geq x \text{ for some } \sigma \in \cup_{k\leq n}\Sigma_n \right\} \\
C_n &= \left\{ \sum_{\sigma\in\Sigma_n} e^{\beta(X_\sigma - \sqrt{2\log 2}n + \frac{3}{2\sqrt{2\log 2}}\log n)} \mathbf{1}\left\{ \max_{k\leq n}(X_{\sigma|k} - \sqrt{2\log 2}k) \leq x \right\} \right. \\
&\qquad \left. \times \mathbf{1}\left\{ \max_{\frac{n}{2}\leq k\leq n}(X_{\sigma|k} - \sqrt{2\log 2}k) \geq x - \frac{3}{2\sqrt{2\log 2}}\log n \right\} \geq \frac{e^{\beta x}}{2} \right\} \\
D_n &= \left\{ \sum_{\sigma\in\Sigma_n} e^{\beta(X_\sigma - \sqrt{2\log 2}n + \frac{3}{2\sqrt{2\log 2}}\log n)} \mathbf{1}\left\{ \max_{k\leq n}(X_{\sigma|k} - \sqrt{2\log 2}k) \leq x \right\} \right. \\
&\qquad \times \mathbf{1}\left\{ \max_{\frac{n}{2}\leq k\leq n}(X_{\sigma|k} - \sqrt{2\log 2}k) \leq x - \frac{3}{2\sqrt{2\log 2}}\log n \right\} \\
&\qquad \left. \times \mathbf{1}\left\{ X_\sigma \leq \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x \right\} \geq \frac{e^{\beta x}}{2} \right\}.
\end{aligned}
$$

To see that this is indeed the case, let c_σ be the summand in C_n and d_σ the summand in D_n. Then we have

$$
\begin{aligned}
\sum_{\sigma\in\Sigma_n}(c_\sigma + d_\sigma) &\geq \mathbf{1}\left\{ \max_{\sigma\in\Sigma_n} X_\sigma \leq \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x \right\} \\
&\quad \times \mathbf{1}\left\{ \max_{\sigma\in\cup_{k\leq n}\Sigma_k}(X_\sigma - \sqrt{2\log 2}|\sigma|) \leq x \right\} \sum_{\sigma\in\Sigma_n} e^{\beta(X_\sigma - \sqrt{2\log 2}n + \frac{3}{2\sqrt{2\log 2}}\log n)} \\
&= \mathbf{1}_{A_n^C}\mathbf{1}_{B_n^C} \exp\left(\beta\left(\frac{1}{\beta}\log \sum_{\sigma\in\Sigma_n} e^{\beta X_\sigma} - \sqrt{2\log 2}n + \frac{3}{2\sqrt{2\log 2}}\log n \right)\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathbf{1}_{\mathcal{F}_n\cap A_n^C\cap B_n^C} &= \mathbf{1}_{A_n^C}\mathbf{1}_{B_n^C}\mathbf{1}\left\{ e^{\beta(\frac{1}{\beta}\log\sum_{\sigma\in\Sigma_n} e^{\beta X_\sigma} - \sqrt{2\log 2}n + \frac{3}{2\sqrt{2\log 2}}\log n)} \geq e^{\beta x} \right\} \\
&\leq \mathbf{1}\left\{ \sum_{\sigma\in\Sigma_n}(c_\sigma + d_\sigma) \geq e^{\beta x} \right\} \\
&\leq \mathbf{1}_{C_n} + \mathbf{1}_{D_n}
\end{aligned}
$$

and

$$
\mathbf{1}_{\mathcal{F}_n} \leq \mathbf{1}_{A_n} + \mathbf{1}_{B_n} + \mathbf{1}_{\mathcal{F}_n\cap A_n^C\cap B_n^C} \leq \mathbf{1}_{A_n} + \mathbf{1}_{B_n} + \mathbf{1}_{C_n} + \mathbf{1}_{D_n}. \tag{6.30}
$$

The desired bound for $\mathbb{P}(A_n)$ follows from Proposition 6.2. The bound for $\mathbb{P}(B_n)$ was also calculated in the proof of Proposition 6.2.

3.1. Estimating $\mathbb{P}(D_n)$. Theorem 2.1 is our main tool in estimating $\mathbb{P}(D_n)$ as well: by Markov's inequality

$$\mathbb{P}(D_n) \leq 2\frac{n^{\frac{3\beta}{2\sqrt{2\log 2}}}}{e^{\beta x}}\mathbb{E}\left(\sum_{\sigma\in\Sigma_n} e^{\beta(X_\sigma-\sqrt{2\log 2}n)}\mathbf{1}\left\{\max_{k\leq n}(X_{\sigma|k}-\sqrt{2\log 2}k)\leq x\right\}\right.$$
$$\times\mathbf{1}\left\{\max_{\frac{n}{2}\leq k\leq n}(X_{\sigma|k}-\sqrt{2\log 2}k)\leq x-\frac{3}{2\sqrt{2\log 2}}\log n\right\}$$
$$\left.\times\mathbf{1}\left\{X_\sigma\leq\sqrt{2\log 2}n-\frac{3}{2\sqrt{2\log 2}}\log n+x\right\}\right)$$
$$=2n^{\frac{3\beta}{2\sqrt{2\log 2}}}e^{-\beta x}\mathbb{E}\left(e^{(\beta-\sqrt{2\log 2})S_n}\mathbf{1}\left\{\max_{k\leq n}S_k\leq x\right\}\right.$$
$$\left.\mathbf{1}\left\{\max_{\frac{n}{2}\leq k\leq n}S_k\leq x-\frac{3}{2\sqrt{2\log 2}}\log n\right\}\mathbf{1}\left\{S_n\leq x-\frac{3}{2\sqrt{2\log 2}}\log n\right\}\right).$$

In terms of Brownian bridges, the expectation is

$$\int_{-\infty}^{x-\frac{3}{2\sqrt{2\log 2}}\log n}dz\frac{e^{-\frac{z^2}{n}}}{\sqrt{\pi n}}\int_{-\infty}^{x-\frac{3}{2\sqrt{2\log 2}}\log n-z}dy\frac{e^{-\frac{y^2}{n}}}{\sqrt{\pi n}}e^{(\beta-\sqrt{2\log 2})(y+z)}$$
$$\times\mathbb{P}^{0,z}_{\frac{n}{2}}\left(Y_k\leq x,\text{ for }k\leq\frac{n}{2}\right)\mathbb{P}^{z,y+z}_{\frac{n}{2}}\left(Y_k\leq x-\frac{3}{2\sqrt{2\log 2}}\log n,\text{ for }k\leq\frac{n}{2}\right)$$
$$=e^{(\beta-\sqrt{2\log 2})x}n^{\frac{3}{2}\left(1-\frac{\beta}{\sqrt{2\log 2}}\right)}\int_{-\infty}^{0}dz\frac{e^{-\frac{(z+x-\frac{3}{2\sqrt{2\log 2}}\log n)^2}{n}}}{\sqrt{\pi n}}$$
$$\int_{-\infty}^{0}dy\frac{e^{-\frac{(y-z+x-\frac{3}{2\sqrt{2\log 2}}\log n)^2}{n}}}{\sqrt{\pi n}}e^{(\beta-\sqrt{2\log 2})y}\mathbb{P}^{-x,z-\frac{3}{2\sqrt{2\log 2}}\log n}_{\frac{n}{2}}\left(Y_k\leq 0,\text{ for }k\leq\frac{n}{2}\right)$$
$$\mathbb{P}^{z,y}_{\frac{n}{2}}\left(Y_k\leq 0,\text{ for }k\leq\frac{n}{2}\right)$$
$$\leq Cxe^{(\beta-\sqrt{2\log 2})x}n^{\frac{3}{2}(1-\frac{\beta}{\sqrt{2\log 2}})}$$
$$\int_{-\infty}^{0}dz\frac{e^{-\frac{(z+x-\frac{3}{2\sqrt{2\log 2}}\log n)^2}{n}}}{\sqrt{\pi n}}n^{-1}\left(-z+\frac{3}{2\sqrt{2\log 2}}\log n\right)(1-z)$$
$$\int_{-\infty}^{0}dy\frac{e^{-\frac{(y-z+x-\frac{3}{2\sqrt{2\log 2}}\log n)^2}{n}}}{\sqrt{\pi n}}e^{(\beta-\sqrt{2\log 2})y}n^{-1}(1-y).$$

In the y-integral, we can estimate the Gaussian part upwards to one and the integral still converges so we have an upper bound of order $n^{-\frac{3}{2}}$.

For the z-integral we note that

$$\begin{aligned}
&\int_{-\infty}^{0} dz \frac{e^{-\frac{(z+x-\frac{3}{2\sqrt{2\log 2}}\log n)^2}{n}}}{\sqrt{\pi n}}(1-z)\left(-z+\frac{3}{2\sqrt{2\log 2}}\log n\right)\\
&\quad\le \int_{-\frac{3}{2\sqrt{2\log 2}}\log n}^{0} dz \frac{e^{-\frac{(z+x-\frac{3}{2\sqrt{2\log 2}}\log n)^2}{n}}}{\sqrt{\pi n}}(1-z)\left(-z+\frac{3}{2\sqrt{2\log 2}}\log n\right)\\
&\quad+ \int_{-\infty}^{-\frac{3}{2\sqrt{2\log 2}}\log n} dz \frac{e^{-\frac{(z+x-\frac{3}{2\sqrt{2\log 2}}\log n)^2}{n}}}{\sqrt{\pi n}}(1-z)\left(-z+\frac{3}{2\sqrt{2\log 2}}\log n\right)\\
&\quad\le Cn^{-\frac{1}{2}}(\log n)^3 + C\int_{-\infty}^{-\frac{3}{2\sqrt{2\log 2}}\log n} dz \frac{e^{-\frac{(z+x-\frac{3}{2\sqrt{2\log 2}}\log n)^2}{n}}}{\sqrt{\pi n}} z^2\\
&\quad\le C + Cn\int_{-\infty}^{0} dz \frac{e^{-\frac{(z+\frac{x}{\sqrt{n}}-\frac{3}{2\sqrt{2\log 2}}\frac{\log n}{\sqrt{n}})^2}{n}}}{\sqrt{\pi}} z^2\\
&\quad\le Cn.
\end{aligned}$$

We conclude that

$$\mathbb{P}(D_n) \le Cxe^{-\sqrt{2\log 2}x}. \tag{6.31}$$

3.2. Estimating $\mathbb{P}(C_n)$. The estimate for $\mathbb{P}(C_n)$ requires a bit more work than the others. The proof is very similar to the proof of Proposition 6.2. We begin by writing

$$
\begin{aligned}
\mathbb{P}(C_n) = \mathbb{P}\Bigg(& \sum_{\sigma\in\Sigma_n} e^{\beta(X_\sigma - \sqrt{2\log 2}n + \frac{3}{2\sqrt{2\log 2}}\log n)} \mathbf{1}\left\{\max_{k\le n}(X_{\sigma|k} - \sqrt{2\log 2}k) \le x\right\} \\
& \times \mathbf{1}\left\{\max_{\frac{n}{2}\le k\le n}(X_{\sigma|k} - \sqrt{2\log 2}k) \ge x - \frac{3}{2\sqrt{2\log 2}}\log n\right\} \\
& \times \mathbf{1}\left\{\max_{\sigma\in\Sigma_n} X_\sigma \ge \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x\right\} \ge \frac{e^{\beta x}}{2}\Bigg) \\
+ \mathbb{P}\Bigg(& \sum_{\sigma\in\Sigma_n} e^{\beta(X_\sigma - \sqrt{2\log 2}n + \frac{3}{2\sqrt{2\log 2}}\log n)} \mathbf{1}\left\{\max_{k\le n}(X_{\sigma|k} - \sqrt{2\log 2}k) \le x\right\} \\
& \times \mathbf{1}\left\{\max_{\frac{n}{2}\le k\le n}(X_{\sigma|k} - \sqrt{2\log 2}k) \ge x - \frac{3}{2\sqrt{2\log 2}}\log n\right\} \\
& \times \mathbf{1}\left\{\max_{\sigma\in\Sigma_n} X_\sigma \le \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x\right\} \ge \frac{e^{\beta x}}{2}\Bigg) \\
\le \mathbb{P}\Bigg(& \max_{\sigma\in\Sigma_n} X_\sigma \ge \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x\Bigg) \\
+ \mathbb{P}\Bigg(& \sum_{\sigma\in\Sigma_n} e^{\beta(X_\sigma - \sqrt{2\log 2}n + \frac{3}{2\sqrt{2\log 2}}\log n)} \mathbf{1}\left\{\max_{k\le n}(X_{\sigma|k} - \sqrt{2\log 2}k) \le x\right\} \\
& \times \mathbf{1}\left\{\max_{\frac{n}{2}\le k\le n}(X_{\sigma|k} - \sqrt{2\log 2}k) \ge x - \frac{3}{2\sqrt{2\log 2}}\log n\right\} \\
& \times \mathbf{1}\left\{X_\sigma \le \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x\right\} \ge \frac{e^{\beta x}}{2}\Bigg).
\end{aligned}
$$

Since we have already estimated the first tirm, it is the second one we need to consider. We introduce some notation for it: let $t = (t_1, ..., t_n)$ be some random vector and consider the events

$$
\begin{aligned}
E_{i,j,k}(t) = \Bigg\{ & \max_{l\le n} t_l \le x,\ t_k = \max_{\frac{n}{2}\le l\le n} t_l \in x - \frac{3}{2\sqrt{2\log 2}}\log n + [i, i+1); \\
& t_n + \frac{3}{2\sqrt{2\log 2}}\log n \in (x-1-j, x-j]\Bigg\}.
\end{aligned}
$$

We introduce these events to split the event into a union of parts where we control the value of X_σ, the value of the maximum of $(X_{\sigma|k} - \sqrt{2\log 2}k)_k$ and at which time k the maximum occurs (note that the upper bound for i follows from the fact that for larger i, the set is empty):

$$
\begin{aligned}
& \Bigg\{\max_{k\le n}(X_{\sigma|k} - \sqrt{2\log 2}k) \le x,\ \max_{\frac{n}{2}\le k\le n}(X_{\sigma|k} - \sqrt{2\log 2}k) \ge x - \frac{3}{2\sqrt{2\log 2}}\log n, \\
& X_\sigma \le \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x\Bigg\} \\
& = \bigcup_{j=0}^{\infty}\bigcup_{k=\frac{n}{2}}^{n}\bigcup_{i=0}^{\frac{3}{2\sqrt{2\log 2}}\log n - 1} E_{i,j,k}\left((X_{\sigma|l} - \sqrt{2\log 2}l)_l\right).
\end{aligned}
$$

We then fix an integer $a \in [\frac{3n}{4}, n]$ (which we shall optimize over later on) and also write

$$F_1^i(t) = \bigcup_{j=0}^{\infty} \bigcup_{k=\frac{n}{2}}^{a} E_{i,j,k}(t) \tag{6.32}$$

and

$$F_2^i(t) = \bigcup_{j=0}^{\infty} \bigcup_{k=a+1}^{n} E_{i,j,k}(t). \tag{6.33}$$

We have two elementary ways to estimate the probability we are interested in: Markov's inequality and noting that under the event $\{X_\sigma \leq \sqrt{2\log 2}n - \frac{3}{2\sqrt{2\log 2}}\log n + x\}$ we can estimate the exponential in the sum upwards to $e^{\beta x}$ and cancel the exponential terms in the probability and resort to random walk estimates. The role of the parameter a is to control which of these estimates works better.

Let $(S_k)_k$ be a random walk with standard Gaussian increments and let us now estimate $\mathbb{P}(E_{i,j,k}(S))$ for $\frac{n}{2} \leq k \leq a$. Using some elementary resoning and the Markov property of the random walk, we see that

$$\begin{aligned}
\mathbb{P}\left(E_{i,j,k}(S)\right) \leq \mathbb{P}\Bigg(& \max_{l\leq k} S_l \leq x, \max_{\frac{n}{2}\leq l\leq k} S_l \leq x - \frac{3}{2\sqrt{2\log 2}}\log n + i + 1, \\
& S_k \in x - \frac{3}{2\sqrt{2\log 2}}\log n + [i, i+1), \\
& (S_n - S_k) + S_k + \frac{3}{2\sqrt{2\log 2}}\log n - x \in (-1-j, -j], \\
& \max_{l\geq k}(S_l - S_k) \leq x - \frac{3}{2\sqrt{2\log 2}}\log n + i + 1 - S_k\Bigg) \\
\leq \mathbb{P}\Bigg(& \max_{l\leq k} S_l \leq x, \max_{\frac{n}{2}\leq l\leq k} S_l \leq x - \frac{3}{2\sqrt{2\log 2}}\log n + i + 1, \\
& S_k \in x - \frac{3}{2\sqrt{2\log 2}}\log n + [i, i+1), \\
& (S_n - S_k) \in (-2-j-i, -j-i], \max_{l\geq k}(S_l - S_k) \leq 1\Bigg) \\
= \mathbb{P}\Bigg(& \max_{l\leq k} S_l \leq x, \max_{\frac{n}{2}\leq l\leq k} S_l \leq x - \frac{3}{2\sqrt{2\log 2}}\log n + i + 1, \\
& S_k \in x - \frac{3}{2\sqrt{2\log 2}}\log n + [i, i+1)\Bigg) \\
\times \mathbb{P}\Bigg(& S_{n-k} \in (-2-j-i, -j-i], \max_{l\leq n-k} S_l \leq 1\Bigg).
\end{aligned}$$

We have already estimated terms of this type in the proof of Proposition 6.2. For the second term, we got the bound (for $n \neq k$)

$$\mathbb{P}\left(S_{n-k} \in (-2-i-j, -i-j], \max_{l\leq n-k} S_l \leq 1\right) \leq C(i+j+1)(n-k)^{-\frac{3}{2}}.$$

For the first term we have the bound

$$C(1+j)\left(\mathbf{1}\left\{\frac{n}{2}\leq k\leq\frac{3n}{4}\right\}\frac{\log n}{n^{\frac{3}{2}}}+\mathbf{1}\left\{k\geq\frac{3n}{4}\right\}n^{-\frac{3}{2}}\right). \tag{6.34}$$

So we conclude that for $\frac{n}{2}\leq k\leq a$ (the estimate holds up to $k=n-1$ but we shall only use it up to a)

(6.35)
$$\mathbb{P}(E_{i,j,k}(S))\leq C(1+i+j)^2(n-k)^{-\frac{3}{2}}\left(\mathbf{1}\left\{\frac{n}{2}\leq k\leq\frac{3n}{4}\right\}\frac{\log n}{n^{\frac{3}{2}}}+\mathbf{1}\left\{\frac{3n}{4}\leq k\leq a\right\}n^{-\frac{3}{2}}\right).$$

Applying this, we find

$$\begin{aligned}
&n^{\frac{3\beta}{2\sqrt{2\log 2}}}e^{-\beta x}\mathbb{E}\left(\sum_{\sigma\in\Sigma_n}e^{\beta(X_\sigma-\sqrt{2\log 2}n)}\mathbf{1}_{F_1^i((X_{\sigma|k}-\sqrt{2\log 2}k)_k)}\right)\\
&=n^{\frac{3\beta}{2\sqrt{2\log 2}}}e^{-\beta x}\mathbb{E}\left(e^{(\beta-\sqrt{2\log 2})S_n}\mathbf{1}_{F_1^i(S)}\right)\\
&\leq Cn^{\frac{3\beta}{2\sqrt{2\log 2}}}e^{-\beta x}e^{(\beta-\sqrt{2\log 2})(x-\frac{3}{2\sqrt{2\log 2}}\log n)}\sum_{j\geq 0}\sum_{k=\frac{n}{2}}^{a}e^{-(\beta-\sqrt{2\log 2})j}\mathbb{P}(E_{i,j,k})\\
&\leq Ce^{-\sqrt{2\log 2}x}(1+i)^2\left(\frac{\log n}{\sqrt{n}}+(n-a)^{-\frac{1}{2}}\right).
\end{aligned}$$

For the second estimate, we shall use

$$\begin{aligned}
&\mathbb{P}\left(\sum_{\sigma\in\Sigma_n}\mathbf{1}_{F_2^i((X_{\sigma|k}-\sqrt{2\log 2}k)_k)}\geq\frac{1}{2}\right)\leq\sum_{k=a+1}^{n}\sum_{j=0}^{\infty}\mathbb{P}\left(\sum_{\sigma\in\Sigma_n}\mathbf{1}_{E_{i,j,k}((X_{\sigma|k}-\sqrt{2\log 2}k)_k)}\geq\frac{1}{2}\right)\\
&\leq\sum_{k=a+1}^{n}\sum_{j=0}^{\infty}\mathbb{P}\Bigg(\max_{l\leq n}(X_{\sigma|l}-\sqrt{2\log 2}l)\leq x,\\
&(X_{\sigma|k}-\sqrt{2\log 2}k)=\max_{\frac{n}{2}\leq l\leq n}(X_{\sigma|l}-\sqrt{2\log 2}l)\in x-\frac{3}{2\sqrt{2\log 2}}\log n+[i,i+1),\\
&X_\sigma-\sqrt{2\log 2}n+\frac{3}{2\sqrt{2\log 2}}\log n\in(x-1-j,x-j],\ \text{for some }\sigma\in\Sigma_n\Bigg)\\
&\leq\sum_{k=a+1}^{n}\mathbb{P}\Bigg(\max_{l\leq k}(X_{\sigma|l}-\sqrt{2\log 2}l)\leq x,\\
&\max_{\frac{n}{2}\leq l\leq k}(X_{\sigma|l}-\sqrt{2\log 2}l)\leq x-\frac{3}{2\sqrt{2\log 2}}\log n+i+1,\\
&X_\sigma-\sqrt{2\log 2}k\in x-\frac{3}{2\sqrt{2\log 2}}\log n+[i,i+1),\ \text{for some }\sigma\in\Sigma_k\Bigg)\\
&\leq\sum_{k=a+1}^{n}\mathbb{E}\Bigg(e^{-\sqrt{2\log 2}S_k}\mathbf{1}\Bigg\{\max_{l\leq k}S_l\leq x,\ \max_{\frac{n}{2}\leq l\leq k}S_l\leq x-\frac{3}{2\sqrt{2\log 2}}\log n+i+1,\\
&S_k\in x-\frac{3}{2\sqrt{2\log 2}}\log n+[i,i+1)\Bigg\}\Bigg).
\end{aligned}$$

Let us write out the expectation explicitly and use our Brownian bridge estimates:

$$\mathbb{E}\Bigg(e^{-\sqrt{2\log 2}S_k}\mathbf{1}\Bigg\{\max_{l\le k} S_l \le x,\ \max_{\frac{n}{2}\le l\le k} S_l \le x-\frac{3}{2\sqrt{2\log 2}}\log n+i+1,$$
$$S_k\in x-\frac{3}{2\sqrt{2\log 2}}\log n+[i,i+1)\Bigg\}\Bigg)$$
$$=\int_{-\infty}^{x-\frac{3}{2\sqrt{2\log 2}}\log n+i+1} dz\frac{e^{-\frac{z^2}{n}}}{\sqrt{\pi n}}\int_{x-\frac{3}{2\sqrt{2\log 2}}\log n+i-z}^{x-\frac{3}{2\sqrt{2\log 2}}\log n+i+1-z} dy\frac{e^{-\frac{y^2}{2(k-\frac{n}{2})}}}{\sqrt{2\pi(k-\frac{n}{2})}}e^{-\sqrt{2\log 2}(y+z)}$$
$$\times\mathbb{P}^{0,z}_{\frac{n}{2}}\left(Y_l\le x,\ \text{for all } l\le\frac{n}{2}\right)\mathbb{P}^{z,y+z}_{k-\frac{n}{2}}\left(Y_l\le x-\frac{3}{2\sqrt{2\log 2}}\log n+i+1 \text{ for all } l\le k-\frac{n}{2}\right)$$
$$\le n^{\frac{3}{2}}e^{-\sqrt{2\log 2}x}e^{-\sqrt{2\log 2}i}\int_{-\infty}^{x-\frac{3}{2\sqrt{2\log 2}}\log n+i+1} dz\frac{e^{-\frac{z^2}{n}}}{\sqrt{\pi n}}\int_{x-\frac{3}{2\sqrt{2\log 2}}\log n+i-z}^{x-\frac{3}{2\sqrt{2\log 2}}\log n+i+1-z} dy\frac{e^{-\frac{y^2}{2(k-\frac{n}{2})}}}{\sqrt{2\pi(k-\frac{n}{2})}}$$
$$\times\mathbb{P}^{-x,z-x}_{\frac{n}{2}}\left(Y_l\le 0,\ \text{for all } l\le\frac{n}{2}\right)$$
$$\times\mathbb{P}^{z-x+\frac{3}{2\sqrt{2\log 2}}\log n-i-1,\,y+z-x+\frac{3}{2\sqrt{2\log 2}}\log n-i-1}_{k-\frac{n}{2}}\left(Y_l\le 0 \text{ for all } l\le k-\frac{n}{2}\right)$$
$$=n^{\frac{3}{2}}e^{-\sqrt{2\log 2}x}e^{-\sqrt{2\log 2}i}\int_{-\infty}^{0} dz\frac{e^{-\frac{(z+x-\frac{3}{2\sqrt{2\log 2}}\log n+i+1)^2}{n}}}{\sqrt{\pi n}}\int_{-1}^{0} dy\frac{e^{-\frac{(y-z+x-\frac{3}{2\sqrt{2\log 2}}\log n+i+1)^2}{2(k-\frac{n}{2})}}}{\sqrt{2\pi(k-\frac{n}{2})}}$$
$$\times\mathbb{P}^{-x,z-\frac{3}{2\sqrt{2\log 2}}\log n+i+1}_{\frac{n}{2}}\left(Y_l\le 0,\ \text{for all } l\le\frac{n}{2}\right)$$
$$\times\mathbb{P}^{z,y}_{k-\frac{n}{2}}\left(Y_l\le 0 \text{ for all } l\le k-\frac{n}{2}\right)$$
$$\le Cn^{\frac{3}{2}}xe^{-\sqrt{2\log 2}x}e^{-\sqrt{2\log 2}i}\left(k-\frac{n}{2}\right)^{-\frac{3}{2}}$$
$$\int_{-\infty}^{0} dz\frac{e^{-\frac{(z+x-\frac{3}{2\sqrt{2\log 2}}\log n+i+1)^2}{n}}}{\sqrt{\pi n}}(1-z)\left(1-z+\frac{3}{2\sqrt{2\log 2}}\log n-i-1\right)n^{-1}.$$

Noting that we consider only $i\le\frac{3}{2\sqrt{2\log 2}}\log n-1$, we have essentially already estimated the z-integral in $\mathbb{P}(D_n)$. Recalling that we are interested in $k\ge a\ge\frac{3n}{4}$, we conclude that the entire expectation is bounded by $Cx^{-\sqrt{2\log 2}x}e^{-\sqrt{2\log 2}i}$. Thus

$$\mathbb{P}\left(\sum_{\sigma\in\Sigma_n}\mathbf{1}_{F_2^i((X_{\sigma|k}-\sqrt{2\log 2}k)_k)}\ge\frac{1}{2}\right)\le(n-a)Cx^{-\sqrt{2\log 2}x}e^{-\sqrt{2\log 2}i}. \tag{6.36}$$

Going back to $\mathbb{P}(C_n)$, we see that

$$\mathbb{P}(C_n)\le\mathbb{P}\left(\max_{\sigma\in\Sigma_n}X_\sigma\ge\sqrt{2\log 2}n-\frac{3}{2\sqrt{2\log 2}}\log n+x\right)$$
$$+Ce^{-\sqrt{2\log 2}x}\sum_{i=0}^{\frac{3}{2\sqrt{2\log 2}}\log n-1}\left((1+i)^2\left(\frac{\log n}{\sqrt{n}}+(n-a)^{-\frac{1}{2}}\right)+(n-a)e^{-\sqrt{2\log 2}i}\right).$$

The trick is now to make a dependent of i: let $n-a=\max(1, ae^{bi})$ for some small enough positive a and b. The i-sum will then be bounded in n and we find $\mathbb{P}(C_n) \leq Cxe^{-\sqrt{2\log 2}x}$ and we are done.

4. The case of multiplicative chaos

We note that the proof of Proposition 6.3 did not rely in any way on the correlations of the random variables (X_σ) once the upper bound for the tail of the maximum is known (our proof for the upper bound of the tail of the maximum made some use of the hierarchical structure in the definition of $B_n^x(\sigma)$ - though a modification of our argument might be possible). In [**24**] similar asymptotics for the tail of the maximum of the two-dimensional discrete Gaussian free field were proved (in fact much sharper estimates were shown). Thus proving similar results for the total mass in the low temperature case for more general correlations could be possible.

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